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OPTIMAL SEARCH FOR MOVING TARGETS IN CONTINUOUS TIME AND SPACE USING CONSISTENT APPROXIMATIONS

by

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We show how to formulate many continuous time-and-space search problems as generalized optimal control problems, where multiple searchers look for multiple targets. Specifically, we formulate problems in which we minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon, and problems in which we maximize the expected number of targets detected. We construct discretization schemes to solve these continuous time-and-space problems, and prove that they are consistent approximations. Consistency ensures that global minimizers, local minimizers, and stationary points of the discretized problems converge to global minimizers, local minimizers, and stationary points, respectively, of the original problems. We also investigate the rate of convergence of algorithms based on discretization schemes as a computing budget tends to infinity. We provide numerical results to show that our discretization schemes are computationally tractable, including examples with three searchers and ten targets. We develop three heuristics for real-time search planning, one based on our discretization schemes, and two based on polynomial fitting methods, and compare the three methods to determine which solution technique would be best suited for use onboard unmanned platforms for automatic route generation for search missions.

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OPTIMAL SEARCH FOR MOVING TARGETS IN CONTINUOUS TIME AND SPACE USING CONSISTENT APPROXIMATIONS

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ABSTRACT

We show how to formulate many continuous time-and-space search problems as generalized optimal control problems, where multiple searchers look for multiple targets. Specifically, we formulate problems in which we minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon, and problems in which we maximize the expected number of targets detected. We construct discretization schemes to solve these continuous time-and-space problems, and prove that they are consistent approximations. Consistency ensures that global minimizers, local minimizers, and stationary points of the discretized problems converge to global minimizers, local minimizers, and stationary points, respectively, of the original problems. We also investigate the rate of convergence of algorithms based on discretization schemes as a computing budget tends to infinity. We provide numerical results to show that our discretization schemes are computationally tractable, including examples with three searchers and ten targets. We develop three heuristics for real-time search planning, one based on our discretization schemes, and two based on polynomial fitting methods, and compare the three methods to determine which solution technique would be best suited for use onboard unmanned platforms for automatic route generation for search missions.

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EXECUTIVE SUMMARY

We develop formulations for continuous time-and-space search problems as generalized optimal control problems, where searchers look for non-evading targets that move in a w-dimensional area of interest, where w is a positive integer. We consider a large class of targets, where we assume the targets follow deterministic trajectories, given some information about their initial states or other parameters. We deal with two categories of targets, one where the targets coordinate their actions and one where the targets operate independently.

We refer to the generalized optimal control problem when the targets coordinate their actions as the general target problem. We provide two different problem formulations for this category. The unconstrained and constrained problems (GTP) and (GTP^c), respectively, are formulated to minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon. The unconstrained and constrained problems (GTP^e) and $(GTP^{c,e})$, respectively, are formulated to maximize the expected number of targets detected during the planning horizon.

We refer to the generalized optimal control problem when the targets operate independently as the independent target problem, and formuate the unconstrained and constrained problems (ITP^p) and $(ITP^{c,p})$, respectively, to minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon. We also formuate the unconstrained and constrained problems (ITP^e) and $(ITP^{c,e})$, respectively, to maximize the expected number of targets detected during the planning horizon. We develop discretization schemes to solve (GTP), (GTP^c), (GTP^e), $(GTP^{c,e})$, (ITP^p), $(ITP^{c,p})$, (ITP^e), and $(ITP^{c,e})$, and show that the resulting finite-dimensional problems are consistent approximations to their infinite dimensional counterparts. Consistency of approximation ensures that global minimizers, local minimizers, and stationary points of the discretized problems converge

to global minimizers, local minimizers, and stationary points, respectively, of the original problems.

We consider a broad class of infinte dimensional optimization problems, which includes (GTP), (GTP^c), (GTP^e), $(GTP^{c,e})$, (ITP^p), $(ITP^{c,p})$, (ITP^e), and $(ITP^{c,e})$. We derive rate of convergence results for discretization methods used to solve problems of this class by expressing the rate of convergence in terms of computational work. We find an upper bound for the rate of convergence by considering a finitely convergent optimization algorithm. The finitely convergent algorithm has no optimization error after a sufficiently large number of iterations, so the upper bound on the rate of convergence represents the best possible rate. We show that both superlinear and linear optimization algorithms are also able to attain this upper bound, and we identify specific discretization policies that achieve this best possible rate.

We provide implementable algorithms based on our discretization schemes along with numerical results to show that the problems (GTP^c), $(ITP^{c,p})$, and $(ITP^{c,e})$ are computationally tractable. For the 90 cases we consider, which include examples with three searchers and ten targets, we are able to compute near-stationary solutions in the range of 3 to 20 hours.

We develop three heuristics for real-time search planning, one based on our discretization schemes, and two based on polynomial fitting methods, and compare the three methods to determine which solution technique would be best suited for use onboard unmanned platforms. Our numerical results indicate that our fixed-precision discretization scheme consistently provides solutions with the best objective values in the range of 40 to 200 seconds, and is therefore the best candidate for use as a real-time search planning method.

I. INTRODUCTION

A. MOTIVATION AND BACKGROUND

The need to protect a High Value Unit (HVU), such as an aircraft carrier or other large capital ship, from small boat attack is an important problem faced by navies around the world. The need to protect HVUs from these types of attack was brought into sharp focus following the attack on the USS Cole (DDG 67) in 2000. The need was further demonstrated during a U.S. Navy war game in 2002, in which adversarial forces used swarms of small boats and aircraft to overwhelm a U.S. invasion fleet (Kahwaji, 2006). There are also indications that smaller countries could resort to an asymmetric strategy involving small boats if they were involved in a naval conflict against a major naval power (Kahwaji, 2006). A key component of any defense against small boat attack involves the search for and detection of potential adversaries. To quote Kahwaji, "Surveillance is key: If the raiders can be tracked as they swarm from their bases, they can be sunk with Rockeye cluster bombs and other munitions" (Kahwaji, 2006). In order to provide the best possible defense against small boat attack, it is clear that optimal utilization of search assets is highly desirable.

Because the searchers' trajectories have a significant impact on the probability of finding the targets within a given time horizon, we would like a way to determine the "best" trajectory for each of the searchers. This is a fundamentally difficult problem due to the nonlinearity and nonconvexity introduced by using probability of detection as the basic performance measure of a search platform. It becomes even more difficult if we consider multiple searchers looking for multiple targets. Given the increasing use of unmanned systems, such as unmanned aircraft systems (UASs) and unmanned surface vehicles (USVs), by militaries around the world, it is clear that there is a need for an automated method for finding optimal search trajectories given some initial intelligence information about potential adversaries. This need serves as

motivation for our work. In this dissertation, we consider several search trajectory optimization problems, including cases with multiple searchers looking for multiple targets. The searchers could be aircraft or surface vessels with sensors designed to detect targets that pose a threat to a HVU. We formulate these search problems as generalized optimal control problems, in continuous time and space. We develop and analyze algorithms based on discretization schemes to solve these problems; these schemes are consistent approximations in the sense of Polak (1997), and the resulting algorithms are guaranteed to converge to stationary solutions.

B. SCOPE OF DISSERTATION

We consider search trajectory optimization problems where searchers look for nonevading targets that move in a w-dimensional area of interest, where w is a positive integer. We limit the scope to searchers that move in continuous time and space, according to dynamics defined by ordinary differential equations. It is worth noting that this includes a large class of searchers, such as many manned and unmanned aircraft as well as many manned and unmanned surface vessels. We consider a large class of targets whose trajectories may not be continuous in time and space. While we only consider non-evading targets, targets are allowed to move intelligently. We allow for the case of targets that have perfect knowledge of the HVU's position at all time, who then determine the trajectories required to strike the HVU in minimum time. We require that the targets' motion is conditionally deterministic, which means that the targets follow deterministic trajectories given realizations of random variables that specify information about their initial states or other parameters. We assume that the probability distribution for each random variable is known, and we consider two situations. The first is when the random variables are dependent. This would be true, for example, if the targets chose to attack cooperatively in a swarm configuration. The second is when the random variables describing one target are independent of

the random variables describing another target. This would represent the case when the targets attack with no level of coordination, other than the desire to strike the same HVU.

We refer to the generalized optimal control problem when the targets coordinate their actions as the general target problem. We provide two different problem formulations for the case when the targets coordinate their actions. The first formulation has corresponding problems (GTP) and (GTP^c), for the unconstrained and control-constrained cases, respectively, whose solutions minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon. The second formulation has corresponding problems (GTP^e) and $(GTP^{c,e})$, for the unconstrained and control-constrained cases, respectively, whose solutions maximize the expected number of targets detected during the planning horizon.

We refer to the generalized optimal control problem when the random variables are independent across targets as the independent target problem. We also provide two different problem formulations for the independent target category. The first formulation has corresponding problems (ITP^p) and $(ITP^{c,p})$, for the unconstrained and control-constrained cases, respectively, whose solutions minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon. The second formulation has corresponding problems (ITP^e) and $(ITP^{c,e})$, for the unconstrained and control-constrained cases, respectively, whose solutions maximize the expected number of targets detected during the planning horizon.

Previous work on solving search problems using optimal control has focused on deriving necessary conditions for optimality (see, for example, Hellman, 1970, 1971, 1972, and Lukka, 1977) in the tradition of Pontryagin, or sufficient conditions for optimality (see, for example, Hibey, 1982 and Ohsumi, 1991) in the tradition of Hamilton, Jacobi, and Bellman. Because it is unclear how to use these approaches to develop consistent approximations that converge to solutions of the original problem, we adopt the method developed by E. Polak (see, for example, Section 4.2 in Polak,

1997) to define optimality conditions. As in Section 4.3 of Polak (1997), we use Euler's method to approximately solve the time-discretized ordinary differential equations governing the searcher dynamics. While it is possible to extend Polak's method to include Runge-Kutta integration methods (see, for example, Schwartz & Polak, 1996), doing so introduces additional complications in both theory and numerical implementation. Because this is the first attempt to utilize Polak's method to derive optimality conditions and solutions for these types of generalized optimal control problems, we use Euler's method throughout this dissertation to approximately solve the time-discretized ordinary differential equations.

The problems we have formulated require spatial integration in addition to integration in time. Spatial integration often requires more than one dimension, making numerical methods used to approximate spatial integrals more computationally expensive than those related to time integration. We use Simpson's integration rule as a higher-order approximation to the spatial integrals due to the fact that it helps limit approximation error, and is relatively simple to handle in both theory as well as implementation.

C. LITERATURE SURVEY

Modern search theory traces its roots to the formation of the United States Navy's Operations Research Group (ORG) during World War II. During the war, Dr. B. O. Koopman and his associates in the ORG worked on improving the way the United States Navy conducted search during anti-submarine warfare operations (Iida et al., 2002). Koopman (1980) is an unclassified and updated version of Koopman (1946), which summarizes the work done by the ORG during the war. Koopman (1980), and indeed much of the literature prior to the 1970s, focuses on stationary targets (Benkoski et al., 1991). Research literature on moving targets can be grouped into two categories as given in Benkoski et al. (1991):

(i) articles which address special types of target motion, described below, that are amenable to analysis and

(ii) articles which build on S. Brown's work on conditioning with stationary targets (see Brown, 1980) by developing general necessary and sufficient optimality conditions for moving-target problems.

In this dissertation, we deal with target motion that is amenable to analysis, so we refer the interested reader to Benkoski et al. (1991) for a comprehensive survey of the literature related to category (ii).

With respect to category (i), there are two special types of target motion that appear in the literature. The first type deals with targets whose motion is Markovian in nature, which means that the target moves in a random manner that can be modeled by a Markov process. Benkoski et al. (1991) again provides a comprehensive literature survey for Markovian target motion studies (until 1990). More recent research (see, for example, Washburn, 1998; Lau, Huang, & Dissanayake, 2008, and Sato & Royset, 2010) focuses on the development of specialized branch-and-bound algorithms for finding an optimal path for the single searcher. In addition, Dell et al. (1996) present an exact procedure (utilizing a branch-and-bound algorithm) as well as six heuristics (local search, expected detection heuristics, genetic algorithms, and moving time-horizon heuristic) for solving the multiple searcher problem.

The second type of special target motion in the literature is when the target's motion is assumed to be conditionally deterministic. The term conditionally deterministic means that the target's trajectory depends on random variables, and if the random variables are given, then the target's position is known for all time. This type of target motion has been investigated by many different researchers. In the case of discrete time, Royset and Sato (2010) presents a convex mixed-integer nonlinear program (MINLP) formulation for a route optimization problem involving multiple searchers who seek to detect one or more moving targets. Royset and Sato (2010) propose two solution approaches for the MINLP based on linearizations, one of which involves using a cutting-plane method.

For the case of continuous time, Stone and Richardson (1974) and Stone (1977) derive necessary and sufficient conditions to optimally allocate search effort to maximize the probability of detection during a given planning horizon. Iida (1989) extends the work of Stone and Richardson to include the problem of finding the optimal search plan which minimizes the expected risk (the expected search cost minus the expected reward). Iida (1989) also derives the closed form of the optimal search plan when the target moves straight from a fixed point and selects its course and speed randomly. Pursiheimo (1976) derives a necessary condition for search plans to be optimal when the probability of detection is to be maximized and expected search time is to be minimized for a continuous time, discrete space model. The approach in Pursiheimo (1976) is noteworthy since it is the first to formulate the search problem for a target with conditionally deterministic motion as an optimal control model. The results in Pursiheimo (1976) are theoretical in nature, however, as no search plans are generated.

Following the work of Pursiheimo, there have been other formulations of the search problem using an optimal control model. Lukka (1977) uses an optimal control model and derives a necessary condition for the optimal search plan assuming conditionally deterministic target motion. Mangel (1981) presents an approximate solution method, referred to as the "ray method," that can be used in conjunction with the necessary conditions given by Lukka (1977) to determine the optimal search plan. Hibey (1982) and Ohsumi (1991) use an optimal control model to find sufficient conditions for the optimal search plan assuming Markovian target motion. Ohsumi (1991) also gives a method that can be used to numerically approximate the optimal search trajectories, with simulation studies assuming Markovian target motion.

It should be noted that Lukka (1977), Mangel (1981), Hibey (1982), and Ohsumi (1991) all use an indirect approach to solving the optimal control problem, employing the calculus of variations to obtain first-order optimality conditions, which result in a boundary-value problem that must be solved. A direct method

can also be used to solve the optimal control problem, which eliminates the need to solve a boundary-value problem. In a direct method, a time-discretization scheme is introduced, and the control problem is then transcribed into a nonlinear optimization problem that can be solved using standard techniques. Wasburn (1990) employs a direct method to construct a tactical decision program, JITTER, which is designed to find an optimized trajectory for a submarine transiting from a given starting location to a given terminal location, while minimizing the total acoustical energy received by n listeners. JITTER utilizes discretization and a steepest-descent method that gradually modifies a user-supplied initial trajectory into an optimized path by making first-order corrections.

A more recent approach, given in Chung et al. (2010), uses a direct method based on Chapter 4 of Polak (1997) to solve an optimal control problem which finds optimized search plans for a target that moves straight down a channel at constant speed. The results in Chung et al. (2010) indicate that the direct method based on Chapter 4 of Polak (1997) can be used to generate optimized search plans for as many as three searchers. Because the direct method based on Chapter 4 of Polak (1997) demonstrates the ability to produce search plans and allows us to develop consistent approximations, we focus on this method in the dissertation.

D. CONTRIBUTIONS

In this dissertation, we extend the work of Chung et al. (2010) in many ways. Our target motion model is more general than that found in Chung et al. (2010), and as a result the framework we develop allows for the formulation of many important continuous time-and-space search problems as generalized optimal control problems. We provide four different examples of problem formulations, namely (GTP), (GTP^e), (ITP^p), and (ITP^e), that can be modeled using our framework. We also develop a min-max formulation that can be used in future studies to find searcher trajectories that minimize the maximum expected time of first detection of all the potential

targets, as well as a model that can extend our detection-based approach to one that includes herding potential threats away from a high value unit. As in Chung et al. (2010), we also use a direct method to solve the generalized optimal control problems based on Chapter 4 of Polak (1997), but we are the first to provide proofs to show that the discretization schemes are consistent approximations to their infinite-dimensional counterparts.

We develop rate of convergence results based on expressing the rate of convergence in terms of computational work rather than the traditional number of iterations or level of discretization. Using this approach, we provide an upper bound on the rate of convergence that can be achieved by any optimization algorithm. We also provide discretization policies for superlinearly and linearly convergent optimization algorithms that achieve this upper bound. In addition, we use the rate of convergence results we obtain to provide insight regarding the choice of numerical method used to approximately solve the differential equations as well as approximate the spatial integration when solving generalized optimal control problems.

We provide implementable algorithms based on our discretization schemes, and use them to produce numerical results to show that our method is computationally tractable. While Chung et al. (2010) considers a single target and as many as three searchers, we give numerical examples that include three searchers looking for as many as ten targets. We also investigate methods to reduce the computational cost necessary to obtain these numerical solutions, including adaptive discretization schemes and heuristics based on polynomial fitting.

E. ORGANIZATION

The remainder of this dissertation is outlined as follows. Chapter II gives the objective function definitions, describes the target motion model, defines the searcher dynamics, and provides preliminary generalized optimal control problem formulations. Chapter III introduces the spaces necessary to complete the problem formulations and

develops consistent approximations for the generalized optimal control problems we consider in the dissertation. In Chapter IV, we develop rate-of-convergence results for different classes of optimization algorithms that can be used to solve generalized optimal control problems. Chapter V provides implementable algorithms and gives our numerical results. In Chapter VI we present conclusions and suggest future research opportunities. The Appendix provides some mathematical background.

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II. SITUATIONAL DESCRIPTION AND PROBLEM FORMULATIONS

We consider the situation depicted in Figure 1, where a High Value Unit (HVU) is operating in a two-dimensional area of interest. The HVU follows a fixed trajectory, $\{x^0(t) \in \mathbb{R}^2 : 0 \leq t \leq T < \infty\}$, during the finite planning horizon $[0, T]$. Without loss of generality, we assume throughout the dissertation that $T = 1$, and therefore our normalized planning horizon is $[0, 1]$. During the transit from $x^0(0)$ to $x^0(1)$, the HVU is under threat of attack from L targets. We assume that the motion of these targets is conditionally deterministic, i.e., the l^{th} target's position, $y^l(t; \alpha)$, is known for all time $t \in [0, 1]$ given the realization of a random vector α which takes values in a compact set $A \subset \mathbb{R}^w$, where w is a positive integer. It should be noted that α will be used to represent both a random vector and its realization, but the nature of α should be clear from the context. We assume that the probability distribution for α is known. In an effort to detect the potential threats to the HVU, there are K searchers operating in the vicinity of the HVU. Our goal is to determine the trajectory, $x^k(t)$, for each of the searchers to follow during the planning horizon that optimizes objective functions we define in Sections II.A and II.B.

There are many different ways to measure the effectiveness of a proposed search plan. In this chapter, and throughout the dissertation, we focus on two types of objective functions. The theory we develop can be extended to include other types of objective functions (see, for example, Chapter VI, Sections B.1 and B.2), but for our analysis and numerical results we concentrate on two types of objective functions: the first type seeks to minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon; the second type seeks to maximize the expected number of distinct targets detected during the planning horizon.

In this chapter, we begin by developing both types of objective functions for the case of statistically dependent targets. The term dependent targets refers to the

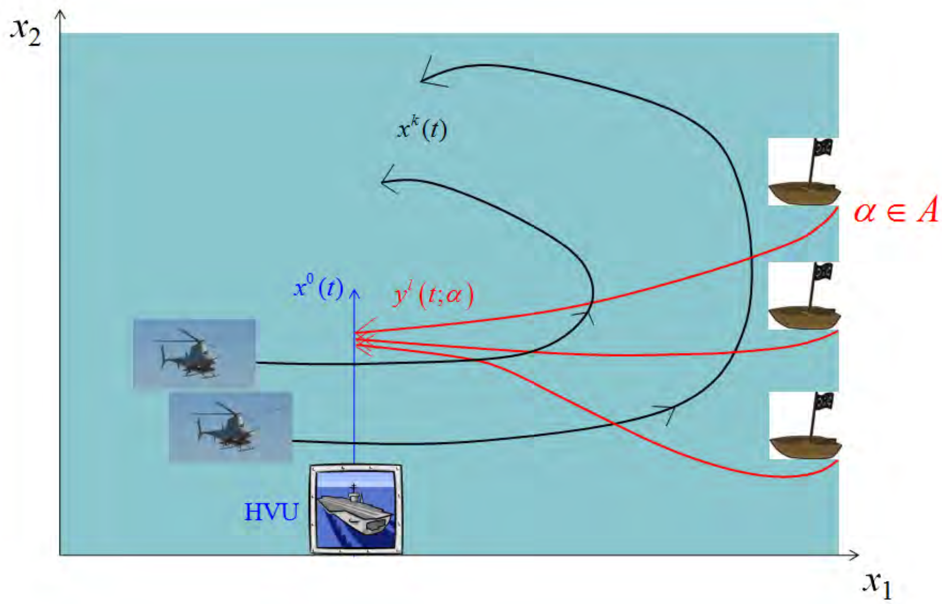

Figure 1. Situational Description.

case when the random variables that define the targets' motion are dependent among targets. A coordinated swarm attack against the HVU would be appropriately modeled by the dependent target case. We then derive both types of objective functions for the case of independent targets. The term independent targets refers to the case when the random variables that define the targets' motion are independent across targets. It should be noted that in the case of independent targets there may be dependence between the random variables that specify information about the parameters for a particular target. An attack by multiple targets whose only level of coordination is the desire to strike the same HVU would be appropriately modeled by the independent target case. Next, we discuss the target motion model. Finally, we give preliminary definitions of the problems we consider in this dissertation. We complete these problem definitions in Chapter III, after we define the appropriate spaces for the searcher control inputs.

A. COORDINATED TARGETS

Given $\alpha \in A \subset \mathbb{R}^w$ common to all targets, the l^{th} target follows a deterministic trajectory, $\left\{y^l(t;\alpha) \in \mathbb{R}^{n_y} : 0 \leq t \leq 1\right\}$, where n_y is a positive integer. As an example, α could be used to represent the uncertainty in target starting location and starting time. Then, given the starting location and starting time for the l^{th} target, some type of deterministic algorithm can be used to find the trajectory for the l^{th} target. While the formulation that follows is valid for any w, the dimension of A, it appears difficult to carry out the analysis for an arbitrary w. This is because w influences the choice and characteristics of the numerical integration scheme used to calculate search plans. During later proofs of convergence it is necessary to quantify the error introduced by the numerical integration scheme, so we now fix $w = 2$.

We let $r^{k,l} : \mathbb{R}^n \times \mathbb{R}^{n_y} \to [0, \infty)$, were n is a positive integer, denote the detection rate for searcher k against target l, which is defined such that $r^{k,l}(x^k, y^l)\Delta t$ approximates the probability of the k^{th} searcher in state $x^k \in \mathbb{R}^n$ detecting the l^{th} target in state $y^l \in \mathbb{R}^{n_y}$ during a small time interval $[t, t + \Delta t)$. The states for the searcher and target typically involve their locations, but could also include other quantities such as heading and time of day. The detection rate reflects the sensor effectiveness and we typically have that $r^{k,l}(x^k, y^l)$ is some decreasing function in the "distance" between x^k and y^l. We put distance in quotations because when $n \neq n_y$, it is necessary to omit certain portions of the state vectors for x^k and y^l in order to properly define a norm that provides a measure of the closeness of the two state vectors. The detection rate, $r^{k,l}(\cdot, \cdot)$, can be selected to reflect various types of sensors and their performance against different types of targets. For theoretical and computational reasons, $r^{k,l}(\cdot, \cdot)$ must satisfy certain differentiability assumptions, which we state in Assumption III.3.

Next, we derive the detection model for a particular α, but first we define some notation. In a manner similar to Chung et al. (2010), given a particular trajectory for the k^{th} searcher $\{x^k(t) : 0 \leq t \leq 1\}$ and a particular trajectory for the l^{th} target

$\{y^l(t;\alpha) : 0 \le t \le 1\}$ under realization α, we denote the probability that the k^{th} searcher does not detect the l^{th} target during $[0,t]$, $t \in [0,1]$, by $q^{k,l}(t;\alpha)$. We assume that events of detection in non-overlapping time intervals are all independent, so we can calculate $q^{k,l}(t;\alpha)$ recursively using the difference equation[1]

$$q^{k,l}(t+\Delta t;\alpha) = q^{k,l}(t;\alpha)\left(1 - \left(r^{k,l}(x^k(t), y^l(t;\alpha))\Delta t + o(\Delta t)\right)\right), q^{k,l}(0;\alpha) = 1, \quad \text{(II.1)}$$

which becomes the parameterized differential equation

$$\dot{q}^{k,l}(t;\alpha) \triangleq \frac{d}{dt}q^{k,l}(t;\alpha) = -q^{k,l}(t;\alpha)r^{k,l}(x^k(t), y^l(t;\alpha)),\ q^{k,l}(0;\alpha) = 1, \quad \text{(II.2)}$$

as $\Delta t \to 0$, with solution

$$q^{k,l}(t;\alpha) = \exp\left(-\int_0^t r^{k,l}(x^k(s), y^l(s;\alpha))ds\right). \quad \text{(II.3)}$$

We assume that the searchers make independent detection attempts and can simultaneously detect multiple targets, and hence it follows from (II.3) that the conditional probability that no searcher detects any target during the time period $[0,1]$, given α and collection of searcher trajectories, $\{x^k(t) : 0 \le t \le 1\}$, $k = 1, 2, ..., K$, is simply the product

$$\begin{aligned}\prod_{k=1}^{K}\prod_{l=1}^{L}\exp\left(-\int_0^1 r^{k,l}(x^k(t), y^l(t;\alpha))dt\right) &= \exp\left(-\sum_{k=1}^{K}\sum_{l=1}^{L}\int_0^1 r^{k,l}(x^k(t), y^l(t;\alpha))dt\right) \\ &= \exp\left(-\int_0^1 \sum_{k=1}^{K}\sum_{l=1}^{L} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right).\end{aligned}$$

Similarly, the conditional probability that no searcher detects the l^{th} target during the time period $[0,1]$, given α and collection of searcher trajectories, $\{x^k(t) : 0 \le t \le 1\}$, $k = 1, 2, ..., K$, is given by the product

$$\prod_{k=1}^{K}\exp\left(-\int_0^1 r^{k,l}(x^k(t), y^l(t;\alpha))dt\right) = \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right).$$

---

[1] Recall that if a function $f : \mathbb{R} \to \mathbb{R}$ is $o(x)$, then $\lim_{x\to 0}\frac{f(x)}{x} = 0$; see for example Definition 5.2 on page 304 in Ross (2007).

The conditional probability that at least one of the searchers detects the l^{th} target during the planning horizon $[0,1]$, given α and collection of searcher trajectories, $\{x^k(t) : 0 \leq t \leq 1\}$, $k = 1, 2, ..., K$, is given by

$$1 - \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right). \tag{II.4}$$

Then, the expected number of targets detected during the time period $[0,1]$, given α and collection of searcher trajectories, $\{x^k(t) : 0 \leq t \leq 1\}$, $k = 1, 2, ..., K$, is given by

$$\sum_{l=1}^{L}\left[1 - \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right)\right]. \tag{II.5}$$

We let $\phi : A \to \mathbb{R}$ be the probability density function of α. For theoretical and computational reasons, $\phi(\cdot)$ must satisfy certain differentiability assumptions which we state in Assumption III.1. Then, the probability that all of the searchers fail to detect any of the targets during $[0,1]$ is given by

$$\int_{\alpha\in A} \exp\left(-\int_0^1 \sum_{k=1}^{K}\sum_{l=1}^{L} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right)\phi(\alpha)d\alpha, \tag{II.6}$$

and the expected number of targets detected during the time period $[0,1]$ is given by

$$\begin{aligned} &\int_{\alpha\in A}\sum_{l=1}^{L}\left[1 - \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right)\right]\phi(\alpha)d\alpha \\ = &\sum_{l=1}^{L}\left[1 - \int_{\alpha\in A}\exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha))dt\right)\phi(\alpha)d\alpha\right]. \end{aligned} \tag{II.7}$$

B. INDEPENDENT TARGETS

We now assume that the random variables that the target motion is conditioned upon are independent among targets. We then have a vector of random variables $\alpha^l \in A \subset \mathbb{R}^w$, one for every target, and we assume that the probability distribution for α^l is known for all $l = 1, 2, ..., L$. Given a particular realization of the random vector, α^l, then the l^{th} target follows a deterministic trajectory, $\left\{y^l(t;\alpha^l) \in \mathbb{R}^{n_y} : 0 \leq t \leq 1\right\}$. We note that the components of α^l can be dependent.

Following the same development as (II.1) through (II.3), with α replaced by α^l, we find that the conditional probability that the k^{th} searcher fails to detect the l^{th} target during the time period $[0, 1]$, given α^l and searcher trajectory $\{x^k(t) : 0 \leq t \leq 1\}$, is given by

$$\exp\left(-\int_0^1 r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right). \tag{II.8}$$

We assume that the searchers make independent detection attempts, and hence it follows from (II.8) that the conditional probability that no searcher detects the l^{th} target, given α^l and collection of searcher trajectories, $\{x^k(t) : 0 \leq t \leq 1\}$, $k = 1, 2, ..., K$, is simply the product

$$\begin{aligned}\prod_{k=1}^{K} \exp\left(-\int_0^1 r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right) &= \exp\left(-\sum_{k=1}^{K}\int_0^1 r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right) \\ &= \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right).\end{aligned}$$

We let $\phi^l : A \to \mathbb{R}$ be the probability density function of α^l, $l = 1, 2, ..., L$, and require it to satisfy certain differentiability assumptions, which we state in Assumption III.31. Then, the probability that all of the searchers fail to detect the l^{th} target during the time period $[0, 1]$ is given by

$$\int_{\alpha^l \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right) \phi^l(\alpha^l)d\alpha^l. \tag{II.9}$$

Hence, the probability that all of the searchers fail to detect any of the targets during $[0, 1]$ is given by

$$\prod_{l=1}^{L}\left[\int_{\alpha^l \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right) \phi^l(\alpha^l)d\alpha^l\right]. \tag{II.10}$$

Based on (II.9), the probability that at least one of the searchers detects the l^{th} target during the planning horizon $[0, 1]$ is given by

$$1 - \int_{\alpha^l \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha^l))dt\right) \phi^l(\alpha^l)d\alpha^l. \tag{II.11}$$

Then, the expected number of targets detected during the time period $[0, 1]$ is given by

$$\sum_{l=1}^{L}\left[1-\int_{\alpha^l\in A}\exp\left(-\int_0^1\sum_{k=1}^{K}r^{k,l}(x^k(t),y^l(t;\alpha^l))dt\right)\phi^l(\alpha^l)d\alpha^l\right]. \tag{II.12}$$

We use (II.6), (II.7), (II.10), and (II.12) to formulate the generalized optimal control problems that we consider in this dissertation, but first we discuss the target motion model which we use to generate the target trajectories $\{y^l(t;\cdot) \in \mathbb{R}^{n_y} : 0 \leq t \leq 1\}$.

C. TARGET MOTION MODEL

We develop this section for the coordinated target case, but note that the independent target case is identical except that α is replaced by α^l. Given a particular α, the target trajectories, $\{y^l(t;\alpha) \in \mathbb{R}^{n_y} : 0 \leq t \leq 1\}$, $l = 1, 2, ..., L$, could be generated in numerous ways. In Chapter III, we make very light assumptions regarding $y^l(\cdot;\cdot)$, so the theory we develop applies to a broad class of targets. In order to provide a specific numerical example, we make the conservative assumption that the targets have full knowledge of the position of the HVU at all time. Based on this information about the HVU, we model the targets as Dubins vehicles, i.e., nonholonomic vehicles that are constrained to move along planar paths of bounded curvature, without reversing direction, that act intelligently and follow trajectories that seek to minimize the time required, t_f, for each of them to hit the HVU. The motion of the targets is subject to additional constraints, which we now discuss in detail.

For any $t \in [0, t_f]$, let $n_y = 3$ and $y^l(t;\alpha) = (y_1^l(t;\alpha), y_2^l(t;\alpha), y_3^l(t;\alpha))^T \in \mathbb{R}^3$, where T denotes the transpose of a vector, be the state of the l^{th} target at time t, with $y_1^l(t;\alpha) \in \mathbb{R}$ and $y_2^l(t;\alpha) \in \mathbb{R}$ denoting the horizontal and vertical components of the location of the l^{th} target, respectively, and $y_3^l(t;\alpha) \in \mathbb{R}$ denoting the heading of the l^{th} target measured from the horizontal axis at time t. For any $t \in [0, t_f]$, the control input, $u^{l,tar}(t)$, for the target is the rate of change of the heading, which is restricted

to the range $[-\bar{u}^{l,tar}, \bar{u}^{l,tar}]$. The target's speed, $\|(\dot{y}_1^l(t;\alpha), \dot{y}_2^l(t;\alpha))\|$, is restricted to the range $[v_{min}^l, v_{max}^l]$. We require that the target's initial and final speeds, v_0^l and v_f^l, be specified. Recall that α is based on a distribution, then given a particular α, the l^{th} target's starting position, $y_0^l(\alpha)$, and starting time, $t_0^l(\alpha)$, are known and we generate its trajectory by approximately solving the following optimal control problem:

$$\min t_f$$

$$\begin{aligned}
\text{s.t.} \quad \dot{y}_1^l(t;\alpha) &= \|(\dot{y}_1^l(t;\alpha), \dot{y}_2^l(t;\alpha))\| \cos y_3^l(t;\alpha), \;\; \forall t \in [t_0^l(\alpha), t_f] && \text{(II.13)} \\
\dot{y}_2^l(t;\alpha) &= \|(\dot{y}_1^l(t;\alpha), \dot{y}_2^l(t;\alpha))\| \sin y_3^l(t;\alpha), \;\; \forall t \in [t_0^l(\alpha), t_f] && \text{(II.14)} \\
\dot{y}_3^l(t;\alpha) &= u^{l,tar}(t), \;\; \forall t \in [t_0^l(\alpha), t_f] && \text{(II.15)} \\
v_{min}^l &\leq \|(\dot{y}_1^l(t;\alpha), \dot{y}_2^l(t;\alpha))\| \leq v_{max}^l, \quad \forall t \in [t_0^l(\alpha), t_f] && \text{(II.16)} \\
-\bar{u}^{l,tar} &\leq u^{l,tar}(t) \leq \bar{u}^{l,tar}, \quad \forall t \in [t_0^l(\alpha), t_f] && \text{(II.17)} \\
v_0^l &= \|(\dot{y}_1^l(t_0^l(\alpha);\alpha), \dot{y}_2^l(t_0^l(\alpha);\alpha))\| && \text{(II.18)} \\
v_f^l &= \|(\dot{y}_1^l(t_f;\alpha), \dot{y}_2^l(t_f;\alpha))\| && \text{(II.19)} \\
y^l(t_0^l(\alpha);\alpha) &= y_0^l(\alpha) && \text{(II.20)} \\
(x_1^0(t_f), x_2^0(t_f)) &= (y_1^l(t_f;\alpha), y_2^l(t_f;\alpha)) && \text{(II.21)}
\end{aligned}$$

The target dynamics are given by equations (II.13), (II.14), and (II.15). Constraint (II.16) restricts the target's speed to the range $[v_{min}^l, v_{max}^l]$. Constraint (II.17) restricts the target's turn rate to the range $[-\bar{u}^{l,tar}, \bar{u}^{l,tar}]$. The boundary conditions given by (II.18) and (II.19) require the target to start and end at the given initial and final speeds, v_0^l and v_f^l. The boundary condition (II.20) requires the target to start at the starting position, $y_0^l(\alpha)$, at the target's starting time, $t_0^l(\alpha)$. The final boundary condition (II.21) requires the target to hit the HVU at the final time, t_f.

We solve this optimal control problem via a direct method that fits seventh-order polynomials to the target trajectories in a manner similar to that found in Yakimenko (2000) and Ghabcheloo et al. (2009). We note that the optimal control problem used to obtain the target trajectories is time-optimal, and we are concerned

with search during the planning horizon $[0, 1]$. This will require truncation of the target trajectories after one hour of travel time, and is further discussed in Section V.B.1.

We are now ready to state the generalized optimal control problems that we consider in this dissertation.

D. GENERALIZED OPTIMAL CONTROL PROBLEMS

Our goal is to find trajectories for the K searchers that optimize the expressions (II.6), (II.7), (II.10), and (II.12). We assume that the motion of the k^{th} searcher, which we refer to as its "physical" dynamics, is governed by the differential equation

$$\dot{x}^k(t) = h^k(x^k(t), u^k(t)), \ t \in [0, 1], \ x^k(0) = \xi^k, \tag{II.22}$$

where ξ^k is the vector of initial conditions for the k^{th} searcher, which could include things such as the initial position and the initial heading of the searcher, the control $u^k(t) \in \mathbb{R}^{m_k}$ is the control input to the k^{th} searcher at time t, which could be the rate of change of the heading of the searcher, and therefore $h^k : \mathbb{R}^n \times \mathbb{R}^{m_k} \to \mathbb{R}^n$. We also define $m = \sum_{k=1}^{K} m_k$, $\xi = (\xi^1, ..., \xi^K)^T$, and $u(t) = (u^1(t)^T, ..., u^K(t)^T)^T$, and therefore the collection of h^k is given by $h : \mathbb{R}^{nK} \times \mathbb{R}^m \to \mathbb{R}^{nK}$. We can also handle time-varying systems and arbitrary planning horizons by using standard transcriptions; see for example p. 493 in Polak (1997).

In order to completely state the generalized optimal control problems, it is necessary to limit the searcher controls to certain spaces which we define in Chapter III. For now, we give preliminary definitions of the generalized optimal control problems.

We define $x^{\eta,k}(t)$, $t \in [0, 1]$, as the solution to (II.22) with $\eta = (\xi, u)$ as the input, and let $x^{\eta}(t) = \left(x^{\eta,1}(t)^T, ..., x^{\eta,K}(t)^T\right)^T$. We assume that the solution $x^{\eta,k}(t)$, $t \in [0, 1]$, exists and is unique based on assumptions we will make in Chapter III. For coordinated targets, we refer to the problem that minimizes the probability that all of the searchers fail to detect any of the targets during $[0, 1]$ by choosing the best η

as the general target problem (GTP). From (II.6), we have the following generalized optimal control problem formulation:

$$(GTP) : \min_{\eta} \left\{ \int_{\alpha \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l}(x^{\eta,k}(t), y^l(t;\alpha))dt \right) \phi(\alpha)d\alpha \right\}. \quad \text{(II.23)}$$

We note that constraints on searcher controls and initial conditions will be given in Chapter III. The constrained general target problem (GTP^c) is exactly the same as (GTP), with the addition of

$$u(t) \in U, t \in [0,1], \quad \text{(II.24)}$$

where U is a convex, compact subset of $\mathbb{R}^m$. For coordinated targets, we refer to the problem that maximizes the expected number of targets detected during the time period $[0,1]$ by choosing the best η as the general target problem (GTP^e). From (II.7), we have the follwing generalized optimal control problem formulation:

$$(GTP^e) : \min_{\eta} \left\{ \sum_{l=1}^{L} \left[1 - \int_{\alpha \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^{\eta,k}(t), y^l(t;\alpha))dt \right) \phi(\alpha)d\alpha \right] \right\}. \quad \text{(II.25)}$$

The constrained general target problem $(GTP^{c,e})$ is exactly the same as (GTP) with the addition of (II.24).

For independent targets, we refer to the problem that minimizes the probability that all of the searchers fail to detect any of the targets during $[0,1]$ by choosing the best η as (ITP^p), and the problem that maximizes the expected number of targets detected during the time period $[0,1]$ by choosing the best η as the independent target problem (ITP^e). From (II.10) and (II.12), respectively, we get the following generalized optimal control problem formulations:

$$(ITP^p) : \min_{\eta} \left\{ \prod_{l=1}^{L} \left[\int_{\alpha^l \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^{\eta,k}(t), y^l(t;\alpha^l))dt \right) \phi^l(\alpha^l)d\alpha^l \right] \right\}, \quad \text{(II.26)}$$

and

$$(ITP^e) : \min_{\eta} \left\{ \sum_{l=1}^{L} \left[1 - \int_{\alpha^l \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} r^{k,l}(x^{\eta,k}(t), y^l(t;\alpha^l))dt \right) \phi^l(\alpha^l)d\alpha^l \right] \right\}, \quad \text{(II.27)}$$

The constrained independent target problems $(ITP^{c,p})$ and $(ITP^{c,e})$ are exactly the same as (ITP^{p}) and (ITP^{e}), respectively, with the addition of (II.24). We again note that constraints on searcher controls and initial conditions for (ITP^{p}) and (ITP^{e}) will be given in Chapter III.

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III. CONSISTENT APPROXIMATIONS

The problems (GTP), (GTP^c), (GTP^e), $(GTP^{c,e})$, (ITP^p), $(ITP^{c,p})$, (ITP^e), and $(ITP^{c,e})$ defined in Chapter II are infinite dimensional in both time and space. In order to numerically solve these problems, some form of discretization is necessary. For the discretization scheme to be useful, it must lead to implementable algorithms that can solve the discretized problems in a reasonable amount of time, and the resulting solutions must correspond to the solutions of the original problems in some sense. The discretization schemes provided in this chapter meet these requirements. In this chapter, we also formally define the term consistent approximation, used to describe the relationship between solutions obtained using finite-dimensional approximating problems to the solutions of the original infinite-dimensional problems.

As alluded to in Chapter II, the problems (GTP), (GTP^c), (GTP^e), $(GTP^{c,e})$, (ITP^p), $(ITP^{c,p})$, (ITP^e), and $(ITP^{c,e})$ are well defined only if the allowable searcher controls are restricted to certain spaces. This chapter begins with specific definitions of those spaces. The remainder of the chapter is then split into two main sections, beginning with the treatment of coordinated targets and followed by independent targets. Both of these main sections proceed in a similar manner, as follows. First, we define an "information state," which we use to write the generalized optimal control problems in terms of the spaces defined in Section III.A. Next, we state our assumptions and define optimality conditions for the generalized optimal control problems. Then, we develop consistent approximations for the time-discretized search problems. Finally, we show that the time- and space-discretized search problems are consistent approximations for the original, continuous time-and-space search problems.

A. CONTROL INPUT

We define the problems (GTP), (GTP^c), (ITP^p), $(ITP^{c,p})$, (ITP^e), and $(ITP^{c,e})$ using the following spaces, which are adopted from Chapters 4 and 5 of

Polak (1997) and described here for the sake of completeness. For the sake of brevity, we do not explicitly deal with the problems (GTP^e) and ($GTP^{c,e}$) in this chapter. We note, however, that results developed in Section III.C for the problems (ITP^e) and ($ITP^{c,e}$) can be trivially extended to include (GTP^e) and ($GTP^{c,e}$), respectively, if α^l is replaced by α and $\phi^l(\cdot)$ is replaced by $\phi(\cdot)$. We assume that the control u is an element of a subset of $L_2^m[0,1]$, the space of Lebesgue square-integrable functions from $[0,1]$ into $\mathbb{R}^m$. Hence, initial condition and control input pairs are elements of the space

$$H_2 \triangleq \mathbb{R}^n \times L_2^m[0,1]. \tag{III.1}$$

We denote the $L_2^m[0,1]$ inner product for any pair of functions $u, v \in L_2^m[0,1]$ by $\langle u, v\rangle_2 \triangleq \int_0^1 \langle u(t), v(t)\rangle\, dt$, with $\langle\cdot,\cdot\rangle$ denoting the Euclidean inner product. The corresponding norms $\|u\|_2 \triangleq \langle u, u\rangle_2^{1/2}$ and $\|\cdot\| \triangleq \langle\cdot,\cdot\rangle^{1/2}$. For any $\eta = (\xi, u) \in H_2$, with $\xi \in \mathbb{R}^n$ and $u \in L_2^m[0,1]$, and any $\eta' = (\xi', u') \in H_2$, with $\xi' \in \mathbb{R}^n$ and $u' \in L_2^m[0,1]$, we define the inner product and norm on H_2, respectively, by

$$\langle \eta, \eta'\rangle_{H_2} \triangleq \langle \xi, \xi'\rangle + \langle u, u'\rangle_2 \tag{III.2}$$

and

$$\|\eta\|_{H_2} \triangleq \left(\|\xi\|^2 + \|u\|_2^2\right)^{1/2}. \tag{III.3}$$

We further restrict the control u to be an element of $L_\infty^m[0,1]$, the space of essentially bounded, measurable functions from $[0,1]$ into $\mathbb{R}^m$, but find it convenient to retain the inner product and norm of $L_2^m[0,1]$. Hence, as in Section 5.6 of Polak (1997), we define the pre-Hilbert space

$$H_{\infty,2} \triangleq \mathbb{R}^n \times L_{\infty,2}^m[0,1], \tag{III.4}$$

where $L_{\infty,2}^m[0,1]$ denotes the space $L_\infty^m[0,1]$ equipped with $\langle\cdot,\cdot\rangle_2$ and $\|\cdot\|_2$. We note that $H_{\infty,2}$ is dense in H_2.

We also assume that there exists a $\rho_{\max} < \infty$ such that controls of interest at all times are located in the interior of $B(0, \rho_{\max}) \triangleq \{v \in \mathbb{R}^m \mid \|v\| \leq \rho_{\max}\}$. Therefore, we focus on the space

$$\mathbf{H} \triangleq \mathbb{R}^n \times \mathbf{U} \subset H_{\infty,2} \tag{III.5}$$

of initial conditions and control inputs, where

$$\mathbf{U} \triangleq \left\{u \in L^m_{\infty,2}[0,1] \mid u(t) \in B(0, \rho_{\max}), \forall t \in [0,1]\right\}. \tag{III.6}$$

When dealing with differentiability statements we restrict ourselves to the subset

$$\mathbf{H}^0 \triangleq \mathbb{R}^n \times \mathbf{U}^0 \subset \mathbf{H}, \tag{III.7}$$

where

$$\mathbf{U}^0 \triangleq \left\{u \in L^m_{\infty,2}[0,1] \mid u(t) \in B(0, \gamma\rho_{\max}), \forall t \in [0,1]\right\}, \tag{III.8}$$

and $\gamma \in (0,1)$ is close to one. It is clear that, when $\rho_{\max} \to \infty$, $\mathbf{H}^0$ "fills" $H_{\infty,2}$.

Finally, we allow for the inclusion of a specific type of pointwise control constraint of the form $u(t) \in U$ for all $t \in [0,1]$, where $U \subset \mathbb{R}^m$ is a convex compact subset of $B(0, \gamma\rho_{\max})$. This means we only consider pointwise control constraints of the form $u \in \mathbf{U}_c$, where

$$\mathbf{U}_c \triangleq \left\{u \in \mathbf{U}^0 \mid u(t) \in U \subset B(0, \gamma\rho_{\max}), \forall t \in [0,1]\right\}. \tag{III.9}$$

We also use the notation

$$\mathbf{H}_c \triangleq \mathbb{R}^n \times \mathbf{U}_c. \tag{III.10}$$

B. COORDINATED TARGETS

1. Information State and Optimal Control Problems

In this section, we define an "information state" which we use to write the generalized optimal control problems in terms of the spaces defined in Section III.A.

For a given $\eta \in \mathbf{H}$, recall that $x^{\eta,k}(t)$, $t \in [0,1]$, is the solution to (II.22) with η as the input. Based on (II.23) we define the objective function $f : \mathbf{H} \to \mathbb{R}$ for any $\eta \in \mathbf{H}$ by

$$f(\eta) \triangleq \int_{\alpha \in A} \exp\left(-\int_0^1 \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l}(x^{\eta,k}(t), y^l(t;\alpha))dt \right) \phi(\alpha) d\alpha. \tag{III.11}$$

In order to simplify the notation in (III.11) as well as facilitate the theoretical development that follows, we find it useful to define a parametric "information state" denoted by $z^\eta(t;\alpha) \in \mathbb{R}$, $t \in [0,1]$, $\alpha \in A$. For any $\alpha \in A$, $t \in [0,1]$, and set of searcher trajectories $\left\{x^{\eta,k}(s), 0 \le s \le t\right\}$, $k = 1, 2, ..., K$, $z^\eta(t;\alpha)$ represents the cumulative detection rate given those searcher trajectories and vector of parameters α and is given by

$$z^\eta(t;\alpha) \triangleq \int_0^t \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l}\left(x^{\eta,k}(s), y^l(s;\alpha)\right) ds, \tag{III.12}$$

or equivalently by the differential equation

$$\dot{z}^\eta(s;\alpha) = \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l}\left(x^{\eta,k}(s), y^l(s;\alpha)\right) \ \forall \ s \in [0,t], \tag{III.13}$$

with $z^\eta(0;\alpha) = 0$. Using this notation, for any $\eta \in \mathbf{H}$, (III.11) simplifies to

$$f(\eta) \triangleq \int_{\alpha \in A} \exp\left(-z^\eta(1;\alpha)\right) \phi(\alpha) d\alpha. \tag{III.14}$$

It is useful to simplify the notation in (III.14) even further. We begin by defining the notation $\tilde{\xi} = \left(\xi^1, ..., \xi^K, 0\right)^T \in \mathbb{R}^{nK+1}$, where $\tilde{\xi}$ is a vector of initial states for the searchers such that the first K elements correspond to the initial "physical" states contained in η and the final element corresponds to the initial "information" state. We then define the function $F : \mathbb{R}^{nK+1} \times \mathbb{R}^{nK+1} \to \mathbb{R}$ such that for any $\tilde{\xi} \in \mathbb{R}^{nK+1}$ and $\tilde{x} \in \mathbb{R}^{nK+1}$, where $\tilde{x} = (\tilde{x}_{-1}, z)^T$, with $\tilde{x}_{-1} \in \mathbb{R}^{nK}$, and $z \in \mathbb{R}$,

$$F(\tilde{\xi}, \tilde{x}) \triangleq e^{-z}. \tag{III.15}$$

To complete our notational simplification, for any $\alpha \in A$, we also define the function $\tilde{f}(\cdot;\alpha) : \mathbf{H} \to \mathbb{R}$ by

$$\tilde{f}(\eta;\alpha) \triangleq F\left(\tilde{\xi}, \tilde{x}^\eta(t;\alpha)\right), \tag{III.16}$$

where $\tilde{x}^\eta(t;\alpha)$ is an augmented state defined by combining the "physical" states with the "information" state as follows

$$\tilde{x}^\eta(t;\alpha) \triangleq \begin{pmatrix} x^\eta(t) \\ z^\eta(t;\alpha) \end{pmatrix} \in \mathbb{R}^{nK+1}, \tag{III.17}$$

where $x^\eta(t) = \left(x^{\eta,1}(t)^T, ..., x^{\eta,K}(t)^T\right)^T$. Using this notation, for any $\eta \in \mathbf{H}$ the objective function in (III.14) simplifies to

$$f(\eta) \triangleq \int_{\alpha \in A} \tilde{f}(\eta;\alpha)\phi(\alpha)d\alpha. \tag{III.18}$$

We now complete the definitions of (GTP) and (GTP^c), which were preliminarily stated in Section II.D. Using the spaces defined in Section III.A we let

$$(GTP) \quad \min_{\eta \in \mathbf{H}^0} \quad f(\eta), \tag{III.19}$$

and

$$(GTP^c) \quad \min_{\eta \in \mathbf{H}_c} \quad f(\eta). \tag{III.20}$$

2. Optimality Conditions

In this section, we state our assumptions and give optimality conditions for (GTP) and (GTP^c). We begin by deriving parameterized differential equations of the augmented dynamics in terms of the augmented state, $\tilde{x}(t;\alpha)$, defined in (III.17). For $t \in [0,1]$ and $\alpha \in A$, we define

$$\tilde{h}(x(t),u(t);\alpha) \triangleq \begin{pmatrix} h^1(x^1(t),u^1(t)) \\ \vdots \\ h^K(x^K(t),u^K(t)) \\ \sum_{k=1}^K \sum_{l=1}^L r^{k,l}(x^k(t),y^l(t;\alpha)) \end{pmatrix} \in \mathbb{R}^{nK+1}, \tag{III.21}$$

where $u(t) = \left(u^1(t)^T, ..., u^K(t)^T\right)^T$. Hence,

$$\dot{\tilde{x}}(t;\alpha) = \tilde{h}(x(t),u(t);\alpha), \;\; t \in [0,1], \tilde{x}(0;\alpha) = \tilde{\xi}. \tag{III.22}$$

We note that $\tilde{x}^\eta(\cdot;\cdot)$ is the solution of (III.22) when the input is $\eta = (\xi, u)$, and the augmented initial conditions are given by $\tilde{\xi} = (\xi^T, 0)^T$. We next state a series of assumptions, beginning with those related to $\phi(\cdot)$ and $y^l(\cdot;\cdot)$. It should be noted that in these assumptions and throughout the dissertation components of vectors are indicated by subscripts on variables.

Assumption III.1. *We assume that $\phi(\cdot)$ is four times continuously differentiable.* □

Assumption III.2. *We assume that*

(i) for all $t \in [0,1]$, $y^l(t;\cdot)$ is four times continuously differentiable for all $l = 1, 2, ..., L$, and

(ii) $y^l(\cdot, \alpha)$ is Lebesgue integrable on $[0,1]$, for all $\alpha \in A$.

□

The compactness of A and Assumptions III.1 and III.2, respectively, ensure that the partial derivatives up to fourth-order of $\phi(\cdot)$ and $y^l(t;\cdot)$, $t \in [0,1]$, are bounded. The assumptions that $\phi(\cdot)$ and $y^l(t;\cdot)$, $t \in [0,1]$, are four times continuously differentiable and the consequence regarding the bounds on their partial derivatives up to fourth-order are sufficient to complete the proofs of convergence in this chapter based on the later decision to use Composite Simpson's rule for our numerical spatial integration scheme; see Section III.B.3b. If we had decided to use another numerical spatial integration scheme, such as Trapezoidal rule, the assumptions on $\phi(\cdot)$ and $y^l(t;\cdot)$, $t \in [0,1]$, could be relaxed to being twice continuously differentiable with the consequence that their partial derivatives up to second-order are bounded. The assumption that $y^l(\cdot, \alpha)$ is Lebesgue integrable on $[0,1]$, for all $\alpha \in A$ is sufficient to ensure that the composite function $r^{k,l}(x^k(\cdot), y^l(\cdot;\alpha))$ is Lebesgue integrable on $[0,1]$, for all $\alpha \in A$, under the assumption that $r^{k,l}(\cdot,\cdot)$ is continuous, and mild assumptions on $x^k(\cdot)$ as given in Assumption III.3; see Azagra et al. (2009).

The next set of assumptions are related to $r^{k,l}(\cdot,\cdot)$ and $\tilde{h}(\cdot,\cdot;\cdot)$. In these assumptions, and throughout the dissertation, we let $h^k_x(\cdot,\cdot)$ denote the $n \times n$ matrix of partial derivatives with element i, j given by $\frac{\partial h^k_i(\cdot,\cdot)}{\partial x_j}$, and let $h^k_u(\cdot,\cdot)$ denote the $n \times m$

matrix of partial derivatives with element (i,j) given by $\frac{\partial h_i^k(\cdot,\cdot)}{\partial u_j}$. We also adopt the matrix norm $\|A\| \triangleq \max_{\|v\|=1} \|Av\|$, for any matrix $A \in \mathbb{R}^{m\times n}$, where $v \in \mathbb{R}^n$ is a column vector. We note that for matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times r}$, and column vector $x \in \mathbb{R}^n$, we have that $\|Ax\| \leq \|A\|\|x\|$, $\|AB\| \leq \|A\|\|B\|$, and $\|xx^T\| = \|x\|^2$ (see for example p. 26 in Gill et al., 1991). We also adopt the notation

$$\tilde{h}_x(x(t),u(t);\alpha) \triangleq \begin{pmatrix} h_x^1(x(t),u(t))^T \\ \vdots \\ h_x^K(x(t),u(t))^T \\ \sum_{k=1}^K \sum_{l=1}^L \nabla_x r^{k,l}(x^k(t),y^l(t;\alpha))^T \end{pmatrix}, \tag{III.23}$$

where $\tilde{h}_x(x(t),u(t);\alpha)$ is a $(nK+1)\times n$ matrix and

$$\tilde{h}_u(x(t),u(t);\alpha) \triangleq \begin{pmatrix} h_u^1(x(t),u(t))^T \\ \vdots \\ h_u^K(x(t),u(t))^T \\ 0 \end{pmatrix}, \tag{III.24}$$

where $\tilde{h}_u(x(t),u(t);\alpha)$ is a $(nK+1)\times m$ matrix.

Assumption III.3. *We assume for all $k = 1,2,...,K$ and $l = 1,2,...,L$ that*

(i) there exists a $C_r < \infty$ such that for all $x^k \in \mathbb{R}^n$ and $y^l \in \mathbb{R}^{n_y}$

$$0 \leq r^{k,l}(x^k,y^l) \leq C_r, \tag{III.25}$$

(ii) $r^{k,l}(\cdot,y^l)$ is continuously differentiable for all $y^l \in \mathbb{R}^{n_y}$,

(iii) $r^{k,l}(x^k,\cdot)$ is four times continuously differentiable for all $x^k \in \mathbb{R}^n$,

(iv) $\nabla_x r^{k,l}(x^k,\cdot)$ is four times continuously differentiable for all $x^k \in \mathbb{R}^n$,

(v) $h^k(\cdot,\cdot)$ is continuously differentiable

(vi) there exist $C_{r1} < \infty$, $C_{r2} < \infty$, and $C_{r3} < \infty$ such that for all $x^k \in \mathbb{R}^n$ and $y^l \in \mathbb{R}^{n_y}$,

$$\left|\frac{\partial^j r^{k,l}(x^k,y^l)}{\partial y_i^j}\right| \leq C_{r1} \quad \forall i = 1,2,...,w,\ j = 1,2,3,4, \tag{III.26}$$

$$\left|\frac{\partial^{j+1} r^{k,l}(x^k,y^l)}{\partial x \partial y_i^j}\right| \leq C_{r2} \quad \forall i = 1,2,...,w,\ j = 1,2,3,4, \tag{III.27}$$

and

$$\|\nabla_x r^{k,l}(x^k, y^l)\| \leq C_{r3}, \tag{III.28}$$

(vii) there exists a constant $\tilde{K} \in [1, \infty)$ such that for all x', $x'' \in \mathbb{R}^n$, v', $v'' \in B(0, \rho_{\max})$, and $\alpha \in A$, the following hold:

$$\|\tilde{h}(x', v'; \alpha) - \tilde{h}(x'', v''; \alpha)\| \leq \tilde{K}\left[\|x' - x''\| + \|v' - v''\|\right], \tag{III.29}$$

$$\|\tilde{h}_x(x', v'; \alpha) - \tilde{h}_x(x'', v''; \alpha)\| \leq \tilde{K}\left[\|x' - x''\| + \|v' - v''\|\right], \tag{III.30}$$

and

$$\|\tilde{h}_u(x', v'; \alpha) - \tilde{h}_u(x'', v''; \alpha)\| \leq \tilde{K}\left[\|x' - x''\| + \|v' - v''\|\right]. \tag{III.31}$$

□

We note that (III.29) implies that there exists a $\tilde{K}' < \infty$ such that for all $x' \in \mathbb{R}^n$, and $v \in B(0, \rho_{\max})$,

$$\|\tilde{h}(x', v)\| \leq \tilde{K}'\left[\|x'\| + 1\right]. \tag{III.32}$$

The assumptions about the detection rate function, $r^{k,l}(\cdot, \cdot)$, in Assumption III.3 are not overly restrictive as they allow for the use of many types of sensor models, and are similar to those used by other researchers (see Chung et al., 2010, for example). Assumptions III.3 (v) and (vii) are standard assumptions that parallel those adopted in Assumption 5.6.2 of Polak (1997). Assumption III.3 (vii) guarantees a unique solution to the differential equations governing the searcher dynamics given by (III.22).

We next show that $f(\cdot)$ is Gateaux differentiable on $\mathbf{H}^0$, but in order to do this we need the following intermediate result.

Lemma III.4. *Suppose that Assumptions III.2 and III.3 are satisfied, then for any $\alpha \in A$, $\eta \in \mathbf{H}^0$, and $\delta\eta \in H_{\infty,2}$,*

(a) $\tilde{f}(\cdot; \alpha)$ has a Gateaux differential $D\tilde{f}(\eta; \alpha; \delta\eta)$ at η and hence a directional derivative $d\tilde{f}(\eta; \alpha; \delta\eta)$, with $d\tilde{f}(\eta; \alpha; \delta\eta) = D\tilde{f}(\eta; \alpha; \delta\eta)$, given by

$$D\tilde{f}(\eta; \alpha; \delta\eta) = \left\langle \nabla_\eta \tilde{f}(\eta; \alpha), \delta\eta \right\rangle_{H_2}, \tag{III.33}$$

where the gradient $\nabla_\eta \tilde{f}(\eta;\alpha) = (\nabla_\xi \tilde{f}(\eta;\alpha), \nabla_u \tilde{f}(\eta;\alpha))^T \in H_{\infty,2}$ is defined by

$$\begin{aligned} \nabla_\xi \tilde{f}(\eta;\alpha) &= p^\eta(0;\alpha), && \text{(III.34)} \\ \nabla_u \tilde{f}(\eta;\alpha)(t) &= \tilde{h}_u(x^\eta(t), u(t);\alpha)^T p^\eta(t;\alpha), \text{for t} \in [0,1], && \text{(III.35)} \end{aligned}$$

and $p^\eta(t;\alpha)$ is the solution of the adjoint equation

$$\dot{p}^\eta(t;\alpha) = -\tilde{h}_x(x^\eta(t), u(t);\alpha)^T p^\eta(t;\alpha), \ t \in [0,1],$$

$$p^\eta(1;\alpha) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\exp\left(-z^\eta(1;\alpha)\right) \end{pmatrix} \in \mathbb{R}^{nK+1}, \tag{III.36}$$

(b) the gradient $\nabla_\eta \tilde{f}(\cdot;\alpha)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$ for all $\alpha \in A$ with Lipschitz constant

$$e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right], \tag{III.37}$$

where $\tilde{K} \in [1,\infty)$ is as in Assumption III.3(vii) and $\tilde{K}' < \infty$ is as in (III.32).

(c) $\tilde{f}(\cdot;\alpha)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$ for all $\alpha \in A$ with Lipschitz constant $\sqrt{2}\tilde{K}e^{\tilde{K}}$, where $\tilde{K} \in [1,\infty)$ is as in Assumption III.3(vii).

(d) $\tilde{f}(\cdot;\alpha)$ has a Frechet differential at η relative to $\mathbf{H}$ that is equal to $D\tilde{f}(\eta;\alpha;\delta\eta)$.

Proof. Parts (a) and (d) result from an application of Corollary 5.6.9 in Polak (1997). Part (b) results from an application of Corollary 5.6.9 and the proof of Lemma 5.6.7 in Polak (1997). For part (c), we deduce from Lemma 5.6.7 in Polak (1997) that $z^\eta(1;\alpha)$ is Lipschitz $\mathbf{H}$-continuous as a function of η with Lipschitz constant $\sqrt{2}\tilde{K}e^{\tilde{K}}$. Since $z^\eta(1;\alpha) \geq 0$ for all $\eta \in \mathbf{H}^0$ and $\alpha \in A$, it follows that $\tilde{f}(\cdot;\alpha)$ is Lipschitz $\mathbf{H}$-continuous with Lipschitz constant $\sqrt{2}\tilde{K}e^{\tilde{K}}$ because the magnitude of the slope of the exponential function with an argument in the domain $(-\infty,0]$ is bounded by one.

□

Proposition III.5. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied. Then, for any $\eta \in \mathbf{H}^0$ and $\delta\eta \in H_{\infty,2}$, $f(\cdot)$ has a Gateaux differential $Df(\eta;\delta\eta)$ at η given by*

$$Df(\eta;\delta\eta) = \langle \nabla f(\eta), \delta\eta \rangle_{H_2}, \tag{III.38}$$

where the gradient $\nabla f(\eta)$ is given by

$$\nabla f(\eta)(t) = \int_{\alpha\in A} \nabla_\eta \tilde{f}(\eta;\alpha)(t)\phi(\alpha)d\alpha, \forall t \in [0,1]. \tag{III.39}$$

Proof. Let $\delta\eta \in H_{\infty,2}$, $\eta \in \mathbf{H}^0$, and $\alpha \in A$ be arbitrary. There exists a $\lambda^* > 0$ such that $\eta + \lambda\delta\eta \in \mathbf{H}$ for all $\lambda \in [0, \lambda^*]$. For $\lambda \in [0, \lambda^*]$, consider the ratio

$$R(\alpha; \eta, \delta\eta, \lambda) \triangleq \frac{\tilde{f}(\eta + \lambda\delta\eta; \alpha) - \tilde{f}(\eta; \alpha)}{\lambda}, \tag{III.40}$$

where from Lemma III.4(a) we know that

$$\lim_{\lambda \downarrow 0} R(\alpha; \eta, \delta\eta, \lambda) = D\tilde{f}(\eta; \alpha; \delta\eta) = \left\langle \nabla \tilde{f}(\eta; \alpha), \delta\eta \right\rangle_{H_2}. \tag{III.41}$$

From Lemma III.4(c) we know that

$$|R(\alpha; \eta, \delta\eta, \lambda)| \leq L\|\delta\eta\|_{H_2}, \tag{III.42}$$

with $L = \sqrt{2}\tilde{K}e^{\tilde{K}}$. We also know that

$$\sqrt{2}\tilde{K}e^{\tilde{K}} \int_{\alpha \in A} \phi(\alpha) d\alpha = \sqrt{2}\tilde{K}e^{\tilde{K}}. \tag{III.43}$$

Then because $\delta\eta$ is a constant with respect to α, it is clear from (III.42) and (III.43) that for all λ, $R(\cdot; \eta, \delta\eta, \lambda)$ is dominated by an integrable function. Then the Lebesgue Dominated Convergence Theorem yields that

$$\lim_{\lambda \downarrow 0} \int_{\alpha \in A} R(\alpha; \eta, \delta\eta, \lambda)\phi(\alpha) d\alpha = \int_{\alpha \in A} \lim_{\lambda \downarrow 0} R(\alpha; \eta, \delta\eta, \lambda)\phi(\alpha) d\alpha. \tag{III.44}$$

We then use (III.44) and (III.41) to obtain that

$$\begin{aligned} Df(\eta; \delta\eta) &= \lim_{\lambda \downarrow 0} \frac{f(\eta + \lambda\delta\eta; \alpha) - f(\eta; \alpha)}{\lambda} \\ &= \lim_{\lambda \downarrow 0} \int_{\alpha \in A} \frac{\tilde{f}(\eta + \lambda\delta\eta; \alpha) - \tilde{f}(\eta; \alpha)}{\lambda} \phi(\alpha) d\alpha \\ &= \lim_{\lambda \downarrow 0} \int_{\alpha \in A} R(\alpha; \eta, \delta\eta, \lambda)\phi(\alpha) d\alpha \\ &= \int_{\alpha \in A} \left\langle \nabla_\eta \tilde{f}(\eta; \alpha), \delta\eta \right\rangle_{H_2} \phi(\alpha) d\alpha. \end{aligned} \tag{III.45}$$

Finally, $\left\langle \nabla_\eta \tilde{f}(\eta; \alpha), \delta\eta \right\rangle_{H_2}$ is linear in $\delta\eta$ because by Lemma III.4(d) $\tilde{f}(\eta; \alpha)$ is Frechet differentiable relative to $\mathbf{H}$. Thus, (III.39) follows directly from (III.45). ◻

Proposition III.5 also holds under weaker assumptions on $\phi(\cdot)$, but because we need Assumption III.1 later we adopt it here as well. This issue regarding Assumption III.1 also applies elsewhere in this chapter. Our next task is to show that $\nabla f(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$.

Lemma III.6. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied, then the gradient $\nabla f(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$.*

Proof. For any $\eta', \eta'' \in \mathbf{H}^0$, and $t \in [0,1]$,

$$\|\nabla f(\eta')(t) - \nabla f(\eta'')(t)\| = \left\| \int_{\alpha \in A} \nabla_\eta \tilde{f}(\eta'; \alpha)(t)\phi(\alpha) d\alpha - \int_{\alpha \in A} \nabla_\eta \tilde{f}(\eta''; \alpha)(t)\phi(\alpha) d\alpha \right\|$$

From Lemma III.4(b) we know for any $\alpha \in A$, $\nabla_\eta \tilde{f}(\cdot; \alpha)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$ with Lipschitz constant $e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right]$. Then,

$$\begin{aligned} &\left\| \int_{\alpha \in A} \nabla_\eta \tilde{f}(\eta'; \alpha)(t)\phi(\alpha) d\alpha - \int_{\alpha \in A} \nabla_\eta \tilde{f}(\eta''; \alpha)(t)\phi(\alpha) d\alpha \right\| \\ &\leq e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right] \|\eta' - \eta''\|_{H_2}. \end{aligned} \tag{III.46}$$

Because

$$\nabla f(\eta)(t) = \begin{pmatrix} \nabla_\xi f(\eta) \\ \nabla_u f(\eta)(t) \end{pmatrix}, t \in [0,1], \tag{III.47}$$

we know from (III.46) that

$$\begin{aligned} &\|\nabla_\xi f(\eta') - \nabla_\xi f(\eta'')\|^2 + \|\nabla_u f(\eta')(t) - \nabla_u f(\eta'')(t)\|^2 \\ &\leq \left[e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right]\right]^2 \|\eta' - \eta''\|_{H_2}^2. \end{aligned} \tag{III.48}$$

Hence,

$$\begin{aligned} &\|\nabla f(\eta') - \nabla f(\eta'')\|_{H_2}^2 \\ = \; & \|\nabla_\xi f(\eta') - \nabla_\xi f(\eta'')\|^2 + \int_0^1 \|\nabla_u f(\eta')(t) - \nabla_u f(\eta'')(t)\|^2 dt \\ \leq \; & \left[e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right]\right]^2 \|\eta' - \eta''\|_{H_2}^2 \\ + \; & \int_0^1 \left[e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right]\right]^2 \|\eta' - \eta''\|_{H_2}^2 dt \\ \leq \; & 2\left[e^{\tilde{K}'}\left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right]\right]^2 \|\eta' - \eta''\|_{H_2}^2. \end{aligned} \tag{III.49}$$

Because $\tilde{K}$ and $\tilde{K}'$ are both non-negative, (III.49) implies

$$\|\nabla f(\eta') - \nabla f(\eta'')\|_{H_2} \leq \sqrt{2}e^{\tilde{K}'} \left[\exp\left(\sqrt{2}\tilde{K}e^{\tilde{K}}\right) + \tilde{K}\left(\sqrt{2}\tilde{K}e^{\tilde{K}} + 1\right)\right] \|\eta' - \eta''\|_{H_2}, \tag{III.50}$$

which completes the proof. □

We adopt the approach of Polak (1997) (see section 4.2), and state our optimality conditions in terms of zeros of optimality functions; see also Section III.B.3a. We specifically define the nonpositive optimality functions $\theta : \mathbf{H}^0 \to \mathbb{R}$ and $\theta^c : \mathbf{H}_c \to \mathbb{R}$ as

$$\theta(\eta) \triangleq -\frac{1}{2}\|\nabla f(\eta)\|^2_{H_2}, \tag{III.51}$$

and

$$\theta^c(\eta) \triangleq \min_{\eta' \in \mathbf{H}_c} \langle \nabla f(\eta), \eta' - \eta \rangle_{H_2} + \frac{1}{2}\|\eta' - \eta\|^2_{H_2}, \tag{III.52}$$

which define optimality conditions for (GTP) and (GTP^c), respectively.

Proposition III.7. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied.*

(a) $\theta(\cdot)$ and $\theta^c(\cdot)$ are $\mathbf{H}^0$-continuous functions.

(b) If $\hat{\eta} \in \mathbf{H}^0$ is a local minimizer of (GTP), then $\theta(\hat{\eta}) = 0$.

(c) If $\hat{\eta} \in \mathbf{H}_c$ is a local minimizer of (GTP^c), then $\theta^c(\hat{\eta}) = 0$.

Proof. The proof follows the same arguments as those for the proof of Theorem 4.2.3 in Polak (1997), with Proposition III.5 taking the place of Corollary 5.6.9 from Polak (1997) and Lemma III.6 taking the place of Theorem 4.1.3 from Polak (1997). □

3. Consistent Approximations

As discussed at the outset of this chapter, in order to solve (GTP) and (GTP^c), some form of discretization is necessary. In this section we define the approximating problems (GTP_N), (GTP^c_N), $(GTP_{NM(N)})$, and $(GTP^c_{NM(N)})$. We also present conditions that must be satisfied to ensure that global minimizers, local minimizers, and stationary points of the approximating problems converge to global minimizers, local minimizers, and stationary points, respectively, of the original problems. We

divide our development into two subsections. Both subsections develop consistent approximations for the pairs $((GTP), \theta)$ and $((GTP^c), \theta_c)$, but the first subsection is a "stepping stone" that only deals with time discretization while the second subsection considers time and space discretization. In both subsections, we adopt the notation from sections 4.3 and 5.6 of Polak (1997).

a. *Time-Discretized Problems*

Let $\mathbb{N}$ denote the positive integers, and let $\mathcal{N}$ be an ordered infinite subset of $\mathbb{N}$ defined by

$$\mathcal{N} \triangleq \left\{2^j\right\}_{j=1}^{\infty}. \tag{III.53}$$

To begin our development, we must first define an infinite set of finite-dimensional subspaces $H_N \subset H_{\infty,2}$, whose union is dense in $H_{\infty,2}$. For $N \in \mathcal{N}$ and $j = 0, 1, .., N-1$, let the functions $\pi_{N,j} : [0,1] \to \mathbb{R}$, be defined by

$$\pi_{N,j}(t) \triangleq \begin{cases} \sqrt{N} & \forall t \in [j/N, \quad (j+1)/N), \quad \text{if} \quad j \leq N-2, \\ \sqrt{N} & \forall t \in [j/N, \quad (j+1)/N], \quad \text{if} \quad j = N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{III.54}$$

We also define the subspace $L_N \subset L^m_{\infty,2}[0,1]$, by

$$L_N \triangleq \left\{ u \in L^m_{\infty,2}[0,1] \;\middle|\; u(\cdot) = \sum_{j=0}^{N-1} \bar{u}_j \pi_{N,j}(\cdot), \bar{u}_j \in \mathbb{R}^m, \forall j = 0, 1, .., N-1 \right\}, \tag{III.55}$$

and let

$$H_N \triangleq \mathbb{R}^n \times L_N \subset H_{\infty,2}. \tag{III.56}$$

The functions $\pi_{N,j}(\cdot)$ form a basis for L_N and are defined such that the relation between H_N and the Euclidean space of coefficients used for numerical computation is isometric. We use the function space H_N for proofs of consistency of approximation, and the Euclidean space of coefficients to develop implementable algorithms. The definition of the Euclidean space of coefficients, $\bar{H}_N$, and further discussion of implementable algorithms are provided in Chapter V. We also define the sets

$$\mathbf{H}_N \triangleq \mathbf{H} \cap H_N, \tag{III.57}$$

$$\mathbf{H}^0_N \triangleq \mathbf{H}^0 \cap H_N, \tag{III.58}$$

and

$$\mathbf{H}_{c,N} \triangleq \mathbf{H}_c \cap H_N, \tag{III.59}$$

with $\mathbf{H}, \mathbf{H}^0$ and $\mathbf{H}_c$ defined in (III.5), (III.7), and (III.10), respectively. Based on the definition of $\mathcal{N}$, it is clear that the subspaces H_N possess a desirable nested structure. This means that for any given $N, N' \in \mathcal{N}$ such that $N' > N, H_N \subset H_{N'}$.

We will make use of Proposition 4.3.1 from Polak (1997), which is included here without proof for the sake of completeness. Below, we use the notation $\to^{\mathcal{N}}$ to indicate convergence of a subsequence defined by $\mathcal{N}$.

Proposition III.8. *(a) Let $\mathbf{H}^0_{cl}$ denote the closure of the set $\mathbf{H}^0$. Then, $\mathbf{H}^0_N \to^{\mathcal{N}} \mathbf{H}^0_{cl}$, as $N \to \infty$, and (b) $\mathbf{H}_{c,N} \to^{\mathcal{N}} \mathbf{H}_c$, as $N \to \infty$, where set convergence is in the sense of Painlevé-Kuratowski[2]; as $N \to \infty$ along the subsequence defined by $\mathcal{N}$.* □

We now consider the approximate solution of (II.22) by means of forward Euler's method. For any $\eta = (\xi, u) \in H_N$ and $N \in \mathcal{N}$, we set $x^{\eta,k}_N(0) = \xi^k$ and for any $j = 0, 1, ..., N-1$, and for all $k = 1, 2, ..., K$

$$x^{\eta,k}_N((j+1)/N) - x^{\eta,k}_N(j/N) = \frac{1}{N} h^k \left(x^{\eta,k}_N(j/N), u^k(j/N) \right). \tag{III.60}$$

Simultaneously, we approximately solve (III.13) also by forward Euler's method. For any $\eta = (\xi, u) \in H_N$, $\alpha \in A$, and $N \in \mathcal{N}$, we set $z^\eta_N(0; \alpha) = 0$, and for any $j = 0, 1, ..., N-1$,

$$z^\eta_N((j+1)/N; \alpha) - z^\eta_N(j/N; \alpha) = \frac{1}{N} \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l} \left(x^{\eta,k}_N(j/N), y^l(j/N; \alpha) \right). \tag{III.61}$$

Using the discretized "information state" given by the recursion (III.61), we define the approximate objective functions $f_N : \mathbf{H}_N \to \mathbb{R}$ for any $\eta \in \mathbf{H}_N$ and $N \in \mathcal{N}$ by

$$f_N(\eta) \triangleq \int_{\alpha \in A} \exp\left(-z^\eta_N(1; \alpha)\right) \phi(\alpha) d\alpha. \tag{III.62}$$

[2]See Definition 5.3.6 in Polak (1997).

Again, for the sake of notational simplification, for any $\alpha \in A$, we also define the functions $\tilde{f}_N(\cdot;\alpha) : \mathbf{H}_N^0 \to \mathbb{R}$ by

$$\tilde{f}_N(\eta;\alpha) \triangleq F\left(\tilde{\xi}, \tilde{x}_N^{\eta}(t;\alpha)\right), \tag{III.63}$$

where $F(\cdot,\cdot)$ is as defined in (III.15) and $\tilde{x}_N^{\eta}(j/N;\alpha)$ is an augmented state defined by

$$\tilde{x}_N^{\eta}(j/N;\alpha) \triangleq \begin{pmatrix} x_N^{\eta}(j/N) \\ z_N^{\eta}(j/N;\alpha) \end{pmatrix} \in \mathbb{R}^{nK+1},\ j = 0, 1, ..., N-1, \tag{III.64}$$

where $x_N^{\eta}(j/N) = \left(x_N^{\eta,1}(j/N)^T, ..., x_N^{\eta,K}(j/N)^T\right)^T$, $j = 0, 1, ..., N-1$. Hence, for any $N \in \mathcal{N}$, we define the following approximating problems

$$(GTP_N) \quad \min_{\eta \in \mathbf{H}_N^0} \quad f_N(\eta), \tag{III.65}$$

and

$$(GTP_N^c) \quad \min_{\eta \in \mathbf{H}_{c,N}} \quad f_N(\eta). \tag{III.66}$$

We note that the problems (GTP_N) and (GTP_N^c) still have spatial integrals that have not been discretized.

We want to show that $f_N(\cdot)$, $N \in \mathcal{N}$, are Gateaux differentiable and that their gradients are Lipschitz continuous. In order to do this, we begin with an intermediate result about the sums and products of Lipschitz continuous functions. We then prove an additional intermediate result which shows that the $\tilde{f}_N(\cdot;\cdot)$ are Gateaux differentiable, and that their gradients are Lipschitz continuous.

Lemma III.9. *Suppose S is a bounded subset of $\mathbf{H}^0$. If $\digamma, G : S \to \mathbf{H}^0$ are Lipschitz $\mathbf{H}$-continuous functions on S, then $\digamma + G$, $c\digamma$ for any $c \in \mathbb{R}$, and $\digamma G$ are also Lipschitz $\mathbf{H}$-continuous functions on S.*

Proof. The proof follows the same arguments as the proof of Theorem 4.6.3(b) in Sohrab (2003). In Theorem 4.6.3(b), however, the functions $\digamma, G$ are defined on a

bounded interval of $\mathbb{R}$ and our functions $\mathcal{F}, G$ are defined on a bounded subset of $\mathbf{H}^0$. Let $L_{\mathcal{F}}$ and L_G be Lipschitz constants for $\mathcal{F}$ and G, respectively, then for any $\eta, \eta' \in S$,

$$\begin{aligned} \|(\mathcal{F}+G)(\eta)-(\mathcal{F}+G)(\eta')\|_{H_2} &\leq \|\mathcal{F}(\eta)-\mathcal{F}(\eta')\|_{H_2}+\|G(\eta)-G(\eta')\|_{H_2} \\ &\leq (L_{\mathcal{F}}+L_G)\|\eta-\eta'\|_{H_2}, \end{aligned} \tag{III.67}$$

and

$$\|(c\mathcal{F})(\eta)-(c\mathcal{F})(\eta')\|_{H_2}=|c|\|\mathcal{F}(\eta)-\mathcal{F}(\eta')\|_{H_2}\leq |c|L_{\mathcal{F}}\|\eta-\eta'\|_{H_2}. \tag{III.68}$$

Since S is a bounded subset and $\mathcal{F}$ and G are both Lipschitz $\mathbf{H}$-continuous on S, there exists $M>0$ such that $\|\mathcal{F}(\eta)\|_{H_2}\leq M$ and $\|G(\eta)\|_{H_2}\leq M$ for all $\eta \in S$. Then for any $\eta, \eta' \in S$ we have

$$\begin{aligned} \|(\mathcal{F}G)(\eta)-(\mathcal{F}G)(\eta')\|_{H_2} &\leq \|G(\eta)\|_{H_2}\|\mathcal{F}(\eta)-\mathcal{F}(\eta')\|_{H_2} \\ &+ \|\mathcal{F}(\eta')\|_{H_2}\|G(\eta)-G(\eta')\|_{H_2} \\ &\leq (ML_{\mathcal{F}}+ML_G)\|\eta-\eta'\|_{H_2}. \end{aligned} \tag{III.69}$$

□

Lemma III.10. *Suppose that Assumptions III.2 and III.3 are satisfied, and that* $N \in \mathcal{N}$, *then for any* $\alpha \in A$,

(a) the function $\tilde{f}_N(\cdot;\alpha) : \mathbf{H}_N^0 \to \mathbb{R}$, is Gateaux differentiable, and the gradient $\nabla \tilde{f}_N(\eta;\alpha) = \Big(\nabla_\xi \tilde{f}_N(\eta;\alpha),\ \nabla_u \tilde{f}_N(\eta;\alpha)\Big) \in H_N$ is given by

$$\nabla_\xi \tilde{f}_N(\eta;\alpha) = p_N^\eta(0;\alpha), \tag{III.70}$$

$$\nabla_u \tilde{f}_N(\eta;\alpha)(t) = \sum_{j=0}^{N-1} \gamma_N^\eta(j/N;\alpha)\pi_{N,j}(t),\ t\in[0,1], \tag{III.71}$$

where

$$\gamma_N^\eta(j/N;\alpha) = \frac{\sqrt{N}}{N}\tilde{h}_u(x_N^\eta(j/N),u(j/N);\alpha)^T p_N^\eta((j+1)/N;\alpha), \quad j=0,1,...,N-1, \tag{III.72}$$

and $p_N^\eta(\cdot;\alpha)$ is determined by the adjoint equation

$$\begin{aligned} & p_N^\eta(j/N;\alpha) - p_N^\eta((j+1)/N;\alpha) \\ = & \frac{1}{N}\tilde{h}_x(x_N^\eta(j/N), u(j/N);\alpha)^T p_N^\eta((j+1)/N;\alpha), \\ j = & 0, 1, ..., N-1, \\ p_N^\eta(1;\alpha) = & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\exp\left(-z_N^\eta(1;\alpha)\right) \end{pmatrix} \in \mathbb{R}^{nK+1}. \end{aligned} \tag{III.73}$$

(b) For any bounded subset S of $\mathbf{H}$, there exists a Lipschitz constant $L_S < \infty$ such that, for any $N \geq 1$, $j = 0, 1, ..., N$, $\eta, \eta' \in S \cap H_N$, and $\alpha \in A$,

$$\left|\tilde{f}_N(\eta;\alpha) - \tilde{f}_N(\eta';\alpha)\right| \leq L_S\|\eta - \eta'\|_{H_2}, \tag{III.74}$$

and

$$\left\|\nabla\tilde{f}_N(\eta;\alpha) - \nabla\tilde{f}_N(\eta';\alpha)\right\|_{H_2} \leq L_S\|\eta - \eta'\|_{H_2}. \tag{III.75}$$

Proof. The proof of part (a) is the same as the proof of Theorem 5.6.20 in Polak (1997), so it is not repeated here. For the proof of part (b), we begin by proving (III.74). From Theorem 5.6.16 in Polak (1997) we deduce that $z_N^\eta(1;\alpha)$ is Lipschitz continuous as a function of η. Since $z_N^\eta(1;\alpha) \geq 0$ for all $\eta \in \mathbf{H}_N^0$ and $\alpha \in A$, $\tilde{f}_N(\cdot;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous because the magnitude of the slope of the exponential function with an argument in the domain $(-\infty, 0]$ is bounded by one.

To show that $\nabla\tilde{f}_N(\cdot;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets, we look at its component parts beginning with (III.70). To show that $\nabla_\xi\tilde{f}_N(\eta;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets we proceed with an induction argument. From (III.70) we see that $\nabla_\xi\tilde{f}_N(\eta;\alpha)$ is equal to the value of the adjoint at time zero, $p_N^\eta(0;\alpha)$. In order to find $p_N^\eta(0;\alpha)$, we use the recursion found in (III.73). We know that $p_N^\eta(1;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous as a function of η for the same reasons that $\tilde{f}_N(\cdot;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous as a function of η. From Assumption III.3(vii) and Theorem 5.6.16 in Polak (1997) we also know that $\tilde{h}_x\left(x_N^\eta\left(t\right), u\left(t\right);\alpha\right)$ is Lipschitz $\mathbf{H}_N$-continuous as a function of η for any $t \in [0, 1]$, and that its Lipschitz constant,

$L_{\tilde{h}_x} < \infty$, is independent of t and N. Suppose that $p_N^\eta((j+1)/N;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous as a function of η and that its Lipschitz constant, $L_p < \infty$, is independent of N. Because $\tilde{h}_x\left(x_N^\eta(t), u(t); \alpha\right)$ and $p_N^\eta((j+1)/N;\alpha)$ are both Lipschitz $\mathbf{H}_N$-continuous for all $\eta \in S$, and S is a bounded subset of $\mathbf{H}$, there exists $M < \infty$ such that $\|\tilde{h}_x\left(x_N^\eta(t), u(t); \alpha\right)\| \leq M$ and $\|p_N^\eta((j+1)/N;\alpha)\| \leq M$. Then, based on (III.73) and Lemma III.9, we have that

$$\begin{aligned}
\left\|p_N^\eta(j/N;\alpha) - p_N^{\eta'}(j/N;\alpha)\right\| &\leq \left\|p_N^\eta((j+1)/N;\alpha) - p_N^{\eta'}((j+1)/N;\alpha)\right\| \\
&+ \frac{1}{N}\left\|\tilde{h}_x(x_N^\eta(j/N), u(j/N);\alpha)^T p_N^\eta((j+1)/N;\alpha)\right. \\
&- \left.\tilde{h}_x(x_N^{\eta'}(j/N), u(j/N);\alpha)^T p_N^{\eta'}((j+1)/N;\alpha)\right\| \\
&\leq L_p\|\eta - \eta'\|_{H_2} + \frac{1}{N}(ML_{\tilde{h}_x} + ML_p)\|\eta - \eta'\|_{H_2} \\
&= L_p\left(1 + \frac{M\left(\frac{L_{\tilde{h}_x}}{L_p} + 1\right)}{N}\right)\|\eta - \eta'\|_{H_2}. \qquad \text{(III.76)}
\end{aligned}$$

By doing another step in the backward recursion, we find that

$$\begin{aligned}
&\left\|p_N^\eta((j-1)/N;\alpha) - p_N^{\eta'}((j-1)/N;\alpha)\right\| \\
\leq\ & L_p\left(1 + \frac{M\left(\frac{L_{\tilde{h}_x}}{L_p} + 1\right)}{N}\right)^2 \|\eta - \eta'\|_{H_2} \\
\leq\ & L_p\left(1 + \frac{M\left(\frac{L_{\tilde{h}_x}}{L_p} + 1\right)}{N}\right)^N \|\eta - \eta'\|_{H_2}. \qquad \text{(III.77)}
\end{aligned}$$

Let $K_M = M\left(\frac{L_{\tilde{h}_x}}{L_p} + 1\right)$. There then exists an $\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$, $\left(1 + \frac{K_M}{N}\right)^N \leq 2e^{K_M}$. Then, for any $N \geq \bar{N}$, we have

$$\left\|p_N^\eta((j-1)/N;\alpha) - p_N^{\eta'}((j-1)/N;\alpha)\right\| \leq 2L_p e^{K_M}\|\eta - \eta'\|_{H_2}. \qquad \text{(III.78)}$$

For values of N smaller than $\bar{N}$, we define

$$\delta_i \triangleq \left(1 + \frac{K_M}{\bar{N} - i}\right)^{\bar{N} - i}, \ i = 1, 2, ..., \bar{N} - 1, \qquad \text{(III.79)}$$

and

$$K_\delta \triangleq \max\{2, \max_{i=1,2,\ldots,\bar{N}-1} \delta_i\}. \tag{III.80}$$

Then,

$$\left\| p_N^{\eta}((j-1)/N;\alpha) - p_N^{\eta'}((j-1)/N;\alpha) \right\| \leq K_\delta L_p e^{K_M} \|\eta - \eta'\|_{H_2}. \tag{III.81}$$

Hence, it follows by induction that $\nabla_\xi \tilde{f}_N(\cdot;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous, and that the Lipschitz constant is independent of N.

Next, we consider $\nabla_u \tilde{f}_N(\cdot;\alpha)$ given in (III.71). From (III.54) we know that $\pi_{N,j}(t)$ is either 0 or $\sqrt{N}$. The $\frac{\sqrt{N}}{N}$ factor in the definition of $\gamma_N^\eta(\cdot)$ in (III.72) combined with the $\pi_{N,j}(t)$ value of 0 or $\sqrt{N}$ ensures that $\nabla_u \tilde{f}_N(\cdot;\alpha)$ is independent of N. From Assumption III.3(vii) and Theorem 5.6.16 in Polak (1997) we find that $\tilde{h}_u\left(x_N^\eta(t), u(t);\alpha\right)$ is Lipschitz $\mathbf{H}$-continuous as a function of η for any $t \in [0,1]$, and that the Lipschitz constant is independent of t and N. Based on the induction argument given in (III.76) through (III.81), we know that $p_N^\eta(\frac{j+1}{N};\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous as a function of η for all $j = 0, 1, ..., N-2$, and that the Lipschitz constant is independent of N. From (III.54) we see that the summation in (III.71) is zero for all terms except where t is between j/N and $(j+1)/N$. For these non-zero terms, $\nabla_u \tilde{f}_N(\cdot;\alpha)$ is composed of products of Lipschitz $\mathbf{H}_N$-continuous functions whose Lipschitz constants are independent of N, so by Lemma III.9, $\nabla_u \tilde{f}_N(\eta;\alpha)$ is Lipschitz $\mathbf{H}_N$-continuous, and its Lipschitz constant is independent of N. This means that both components of $\nabla \tilde{f}_N(\eta;\alpha)$ are Lipschitz $\mathbf{H}_N$-continuous and their Lipschtiz constants are independent of N and α, so the proof is complete. $\square$

Proposition III.11. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied, and $N \in \mathcal{N}$, then for any $\eta \in \mathbf{H}_N^0$ and $\delta\eta \in H_{\infty,2}$, $f_N(\cdot)$ has a Gateaux differential $Df_N(\eta;\delta\eta) = \langle \nabla f_N(\eta), \delta\eta \rangle_{H_2}$, where*

$$\nabla f_N(\eta)(t) = \int_{\alpha \in A} \nabla_\eta \tilde{f}_N(\eta;\alpha)(t)\phi(\alpha)d\alpha, \forall t \in [0,1]. \tag{III.82}$$

Proof. The proof follows the same arguments as the proof of Proposition III.5 with $\tilde{f}(\cdot;\cdot)$ replaced by $\tilde{f}_N(\cdot;\cdot)$, so it is not repeated here. $\square$

Next, we show that $\nabla f_N(\cdot)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}_N^0$.

Lemma III.12. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied, then the gradient $\nabla f_N(\cdot)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}_N^0$.*

Proof. The proof follows the same arguments as the proof of Lemma III.6 with $\tilde{f}(\cdot;\cdot)$ replaced by $\tilde{f}_N(\cdot;\cdot)$, so it is not repeated here. $\square$

As in Section III.B.2, we state our optimality conditions in terms of zeros of optimality functions. For any $N \in \mathcal{N}$, we define nonpositive optimality functions $\theta_N : \mathbf{H}_N^0 \to \mathbb{R}$ and $\theta_N^c : \mathbf{H}_{c,N} \to \mathbb{R}$ by

$$\theta_N(\eta) \triangleq -\frac{1}{2}\|\nabla f_N(\eta)\|_{H_2}^2, \tag{III.83}$$

and

$$\theta_N^c(\eta) \triangleq \min_{\eta' \in \mathbf{H}_{c,N}} \langle \nabla f_N(\eta), \eta' - \eta \rangle_{H_2} + \frac{1}{2}\left\|\eta' - \eta\right\|_{H_2}^2, \tag{III.84}$$

which characterizes stationary points of (GTP_N) and (GTP_N^c), respectively.

Proposition III.13. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied.*

(a) $\theta_N(\cdot)$ and $\theta_N^c(\cdot)$ are $\mathbf{H}_N^0$-continuous functions.

(b) If $\hat{\eta} \in \mathbf{H}_N^0$ is a local minimizer of (GTP_N), then $\theta_N(\hat{\eta}) = 0$.

(c) If $\hat{\eta} \in \mathbf{H}_{c,N}$ is a local minimizer of (GTP_N^c), then $\theta_N^c(\hat{\eta}) = 0$.

Proof. The proof follows the same arguments as the proof of Proposition 1.1.6 in Polak (1997), with the norms and inner products replaced with their H_2 equivalents. $\square$

We are now ready to develop proofs for consistency of approximation for the pairs $((GTP_N), \theta_N)$ in the sequence $\{((GTP_N), \theta_N)\}_{N \in \mathcal{N}}$. Because we deal with more than one type of problem and its corresponding approximation, it is simpler to define consistent approximations and epi-convergence using abstract problems. We adopt the notation of Section 3.3.1 in Polak (1997) and let $\mathcal{B}$ be a normed linear space, with norm $\|\cdot\|_{\mathcal{B}}$, and let $\bar{\mathbb{R}} \triangleq \mathbb{R} \cup \{-\infty, +\infty\}$. We then define the problem

$$(P) \quad \min_{x \in X} \quad f(x), \tag{III.85}$$

where $f : \mathcal{B} \to \bar{\mathbb{R}}$ is lower semicontinuous and $X \subset \mathcal{B}$ is a constraint set. Then let $\{\mathcal{B}_N\}_{N \in \mathcal{N}}$ be a family of finite-dimensional subspaces of $\mathcal{B}$ such that $\mathcal{B}_{N_1} \subset \mathcal{B}_{N_2}$, for all $N_1 < N_2 \in \mathcal{N}$, and $\cup_{N \in \mathcal{N}} \mathcal{B}_N$ is dense in $\mathcal{B}$. For all $N \in \mathbb{N}$, let $f_N : \mathcal{B}_N \to \bar{\mathbb{R}}$ be a lower semicontinuous function that approximates $f(\cdot)$ on $\mathcal{B}_N$, and let $X_N \subset \mathcal{B}_N$ be an approximation to X. We then define the family of approximating problems

$$(P_N) \quad \min_{x \in X_N} \quad f_N(x), \; N \in \mathcal{N}. \tag{III.86}$$

Finally, we let X_{cl} denote the closure of X and we define the problem (P_{cl}) by

$$(P_{cl}) \quad \min_{x \in X_{cl}} \quad f(x), \tag{III.87}$$

which may not be epi-graphically equivalent to the problem (P), but which we will assume is equivalent to (P) in the sense that it has the same local and global minimizers. Before we define consistent approximations, we must first define what it means for a function to be an optimality function. As on page 398 of Polak (1997), we define an *optimality function* as follows:

Definition III.1. Let S be a subset of $\mathcal{B}$ such that $X \subset S$. We will say that a function $\theta : S \to \mathbb{R}$ is an optimality function for (P) if

(i) $\theta(\cdot)$ is sequentially upper semi-continuous,

(ii) $\theta(x) \leq 0$ for all $x \in S$, and

(iii) if $\hat{x}$ is a local minimizer of (P), then $\theta(\hat{x}) = 0$.

Similarly, for all $N \in \mathcal{N}$, let S_N be a subset of $\mathcal{B}_N \cap S$ such that $X_N \subset S_N \subset S$. We will say that a function $\theta_N : S_N \to \mathbb{R}$ is an optimality function for (P_N) if

(i) $\theta_N(\cdot)$ is sequentially upper semi-continuous,

(ii) $\theta_N(x) \leq 0$ for all $x \in S_N$, and

(iii) if $\hat{x}_N$ is a local minimizer of (P_N), then $\theta_N(\hat{x}_N) = 0$. □

Then, as on page 399 of Polak (1997), we define *consistent approximations* as follows:

Definition III.2. Consider the problems (P), (P_N), and (P_{cl}) defined in (III.85), (III.86), and (III.87), respectively. Let S be a subset of $\mathcal{B}$ such that $X \subset S$ and, in addition, $X_N \subset S$ for all $N \in \mathcal{N}$. Next, for all $N \in \mathcal{N}$, let S_N be a subset of $S \cap \mathcal{B}_N$ such that $X_N \subset S_N \subset S$. Finally, let $\theta^a : S \to \mathbb{R}$ and $\theta_N^a : S_N \to \mathbb{R}$, $N \in \mathcal{N}$, be optimality functions for (P) and (P_N), respectively. We say that the pairs $((P_N), \theta_N^a)$, in the sequence $\{(P_N, \theta_N^a)\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((P), \theta^a)$ if

(i) either (P_N) epi-converges to (P) or (P_N) epi-converges to (P_{cl}), as $N \to \infty$, and

(ii) for any infinite sequence $\{x_N\}_{N\in K}, K \subset \mathcal{N}$, with $x_N \in S_N$ for all $N \in K$, such that $x_N \to x$ as $N \to \infty$, $\limsup_{N\to\infty} \theta_N^a(x_N) \le \theta^a(x)$. □

We base our proofs of epi-convergence on satisfying the conditions of Theorem 3.3.2 in Polak (1997), which we state here without proof for the sake of completeness.

Proposition III.14. *The epigraphs E_N, $N \in \mathcal{N}$, of the problems (P_N) converge to the epigraph E of the problem (P) if and only if*

(a) for every $x \in X$, there exists a sequence $\{x_N\}_{N\in\mathcal{N}}$, with $x_N \in X_N$, such that $x_N \to^{\mathcal{N}} x$, as $N \to \infty$, and $\limsup f_N(x_N) \le f(x)$; and

(b) for every infinite sequence $\{x_N\}_{N\in K}$, with $K \subset \mathcal{N}$, such that $x_N \in X_N$, for all $N \in K$, and $x_N \to^K x$, as $N \to \infty$, $x \in X$ and $\liminf f_N(x_N) \ge f(x)$. □

We would like to establish epi-convergence of (GTP_N) to (GTP); unfortunately, this is not possible. For this reason, we make a small modification to (GTP) and replace $\mathbf{H}^0$ with its closure, $\mathbf{H}^0_{cl}$. We define this new problem as follows

$$(GTP_{cl}) \quad \min_{\eta\in\mathbf{H}^0_{cl}} \quad f(\eta). \tag{III.88}$$

It is possible to establish epi-convergence of (GTP_N) to (GTP_{cl}). We use the problem (GTP_{cl}), with the following assumption.

Assumption III.15. *We assume that all local and global minimizers of (GTP_{cl}) are in $\mathbf{H}^0$.* □

Using an approach similar to that found in Section 3.3 of Polak (1997), we next show that the pairs $((GTP_N), \theta_N)$ in the sequence $\{((GTP_N), \theta_N)\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((GTP_{cl}), \theta)$, which ensures that globally and

locally optimal points as well as stationary points of (GTP_N) converge to corresponding points of (GTP_{cl}), as $N \to \infty$. In order to show the epi-convergence of (GTP_N) to (GTP_{cl}), we will need the following intermediate result.

Proposition III.16. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied. Then, given a bounded subset $S \subset \mathbf{H}$, there exists a $K_S < \infty$ such that for every $\eta \in S \cap \mathbf{H}_N$, and $N \in \mathcal{N}$,*

$$|f_N(\eta) - f(\eta)| \leq \frac{K_S}{N}. \tag{III.89}$$

Proof. For any $\eta \in S \cap \mathbf{H}_N$ and $N \in \mathcal{N}$, we have that

$$\begin{aligned} |f_N(\eta) - f(\eta)| &= \left| \int_{\alpha \in A} \exp\left(-z_N^\eta(1;\alpha)\right) \phi(\alpha) d\alpha - \int_{\alpha \in A} \exp\left(-z^\eta(1;\alpha)\right) \phi(\alpha) d\alpha \right| \\ &\leq \int_{\alpha \in A} \left|\exp\left(-z_N^\eta(1;\alpha)\right) - \exp\left(-z^\eta(1;\alpha)\right)\right| \phi(\alpha) d\alpha. \end{aligned} \tag{III.90}$$

By Theorem 5.6.23 in Polak (1997), we deduce that we can bound the approximation error $|z^\eta(1;\alpha) - z_N^\eta(1;\alpha)|$. Specifically, for any $\alpha \in A$, given a bounded subset $S \subset \mathbf{H}$, for every $\eta \in S \cap \mathbf{H}_N$, and $N \in \mathcal{N}$, we have

$$|z^\eta(1;\alpha) - z_N^\eta(1;\alpha)| \leq \frac{K_S}{N}, \tag{III.91}$$

with

$$K_S \triangleq \max\left\{1, \frac{\tilde{K} K_\xi}{2}\right\} + K_\xi, \tag{III.92}$$

and

$$K_\xi \triangleq \tilde{K}' \exp(\tilde{K}') \ \sup\left\{\|\xi'\| + 1 \mid (\xi', u) \in S\right\}, \tag{III.93}$$

$\tilde{K}$ as in Assumption III.3(vii), and $\tilde{K}'$ as in (III.32). Hence, by the properties of the exponential function and the fact that $z_N^\eta(1;\alpha) \geq 0$ for all $\eta \in \mathbf{H}^0$ and $z^\eta(1;\alpha) \geq 0$ for all $\eta \in \mathbf{H}_N^0$, for any $\alpha \in A$, and every $\eta \in S \cap \mathbf{H}_N$, and $N \in \mathcal{N}$,

$$\left|\exp\left(-z_N^\eta(1;\alpha)\right) - \exp\left(-z^\eta(1;\alpha)\right)\right| \leq |z^\eta(1;\alpha) - z_N^\eta(1;\alpha)| \leq \frac{K_S}{N}. \tag{III.94}$$

Because K_S does not depend on α, we have that

$$|f_N(\eta) - f(\eta)| \leq \int_{\alpha \in A} \left|\exp\left(-z_N^\eta(1;\alpha)\right) - \exp\left(-z^\eta(1;\alpha)\right)\right| \phi(\alpha) d\alpha \leq \frac{K_S}{N}. \tag{III.95}$$

$\square$

We now show that (GTP_N) epi-converges to (GTP_{cl}).

Theorem III.17. *Suppose that Assumptions III.1, III.2, III.3 and III.15 are satisfied. Then (GTP_N) epi-converges to (GTP_{cl}), as $N \to \infty$.*

Proof. We show that (GTP_N) epi-converges to (GTP_{cl}), as $N \to \infty$, by showing that the conditions in Proposition III.14 are satisfied. Let $\eta \in \mathbf{H}^0_{cl}$ be arbitrary. Then, from Proposition III.8, there exists a sequence $\{\eta_N\}_{N\in\mathcal{N}}$ such that $\eta_N \in \mathbf{H}^0_N$, for all $N \in \mathcal{N}$, and $\eta_N \to^{\mathcal{N}} \eta$, as $N \to \infty$. Let $\epsilon > 0$. By the $\mathbf{H}$-continuity of $f(\cdot)$, there exists an $\bar{N} \in \mathcal{N}$ such that for all $N \geq \bar{N}$, $\frac{K_S}{N} \leq \frac{\epsilon}{2}$ and $|f(\eta_N) - f(\eta)| \leq \frac{\epsilon}{2}$, where K_S is as in (III.92). Hence, by Proposition III.16

$$\begin{aligned} |f_N(\eta_N) - f(\eta)| &\leq |f_N(\eta_N) - f(\eta_N)| + |f(\eta_N) - f(\eta)| \\ &\leq \frac{K_S}{N} + \frac{\epsilon}{2} \\ &\leq \epsilon, \end{aligned} \tag{III.96}$$

for all $N \geq \bar{N}$, $N \in \mathcal{N}$. Consequently, $f_N(\eta_N) \to^{\mathcal{N}} f(\eta)$ as $N \to \infty$, satisfying condition (a) of Proposition III.14.

In order to show that condition (b) of Proposition III.14 is satisfied, suppose that a sequence $\{\eta_N\}_{N\in\mathcal{N}}$ is such that $\eta_N \in \mathbf{H}^0_N$ for all $N \in \mathcal{N}$, and $\eta_N \to^{\mathcal{N}} \eta$, as $N \to \infty$. Then, based on the construction of $\mathbf{H}^0_N$ in (III.58), we must have that $\eta \in \mathbf{H}^0_{cl}$. It again follows from the $\mathbf{H}$-continuity of $f(\cdot)$ and Proposition III.16 that $f_N(\eta_N) \to^{\mathcal{N}} f(\eta)$ as $N \to \infty$, which satisfies condition (b) of Proposition III.14. This proves that (GTP_N) epi-converges to (GTP_{cl}). □

In order to show that the pairs $((GTP_N),\ \theta_N)$, in the sequence $\{((GTP_N), \theta_N)\}_{N\in\mathcal{N}}$, and the pairs $((GTP^c_N),\ \theta^c_N)$, in the sequence $\{((GTP^c_N), \theta^c_N)\}_{N\in\mathcal{N}}$ are consistent approximations for the pairs $((GTP_{cl}), \theta)$ and $((GTP^c), \theta^c)$, respectively, we will need the following two results.

Proposition III.18. *Suppose that Assumptions III.1, III.2 and III.3 are satisfied, then for every bounded subset $S \subset \mathbf{H}^0$, there exists a constant $K_F < \infty$ such that, for any $N \in \mathcal{N}$ and $\eta \in S \cap H_N$*

$$\|\nabla f_N(\eta) - \nabla f(\eta)\|_{H_2} \leq \frac{K_F}{N}. \tag{III.97}$$

Proof. Based on the definitions of $f(\eta)$ and $f_N(\eta)$ given in (III.14) and (III.62), respectively, and Propositions III.5 and III.11,

$$\left\| \int_{\alpha \in A} \nabla_\eta \tilde{f}_N(\eta; \alpha)\phi(\alpha)d\alpha - \int_{\alpha \in A} \nabla_\eta \tilde{f}(\eta; \alpha)\phi(\alpha)d\alpha \right\|_{H_2} \leq \int_{\alpha \in A} \left\| \nabla_\eta \tilde{f}_N(\eta; \alpha) - \nabla_\eta \tilde{f}(\eta; \alpha) \right\|_{H_2} \phi(\alpha)d\alpha. \tag{III.98}$$

We now focus on $\left\| \nabla_\eta \tilde{f}_N(\eta; \alpha) - \nabla_\eta \tilde{f}(\eta; \alpha) \right\|_{H_2}$. By Theorem 5.6.26 in Polak (1997), for any $\alpha \in A$, and for every bounded subset $S \subset \mathbf{H}^0$, there exists a $K_F < \infty$ such that, for all $N \in \mathbb{N}$, and $\eta \in S \cap H_N$,

$$\left\| \nabla_\eta \tilde{f}_N(\eta; \alpha) - \nabla_\eta \tilde{f}(\eta; \alpha) \right\|_{H_2} \leq \frac{K_F}{N}. \tag{III.99}$$

We deduce from the proof of Theorem 5.6.26 in Polak (1997) and Assumption III.3 that K_F is independent of α. Then, we have that

$$\int_{\alpha \in A} \left\| \nabla_\eta \tilde{f}_N(\eta; \alpha) - \nabla_\eta \tilde{f}(\eta; \alpha) \right\|_{H_2} \phi(\alpha)d\alpha \leq \frac{K_F}{N} \cdot 1, \tag{III.100}$$

which completes the proof. $\square$

Because we can only establish epi-convergence of (GTP_N) to (GTP_{cl}), and not (GTP_N) to (GTP), we will need the following intermediate result.

Lemma III.19. *Suppose $\mathbf{H}^0_{cl}$ denotes the closure of $\mathbf{H}^0$. If a function $\nabla\beta : \mathbf{H}^0 \to \mathbf{H}$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$, then it is also Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0_{cl}$.*

Proof. Because $\mathbf{H}^0$ is constructed from $\mathbf{H}$ by choosing $\gamma \in (0, 1)$ (see (III.7) and (III.8)), it is always possible to construct an $\mathbf{H}^{0\prime}$ that is slightly larger than $\mathbf{H}^0$ by choosing a γ closer to one. Since $\mathbf{H}^{0\prime}$ contains $\mathbf{H}^0_{cl}$, and it can be shown that $\nabla\beta(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^{0\prime}$, we can conclude that $\nabla\beta(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0_{cl}$. $\square$

We now show that the pairs $((GTP_N), \theta_N)$ in the sequence $\{((GTP_N), \theta_N)\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((GTP_{cl}), \theta)$.

Theorem III.20. *Suppose that Assumptions III.1, III.2, III.3, and III.15 are satisfied, and that* (GTP_{cl}), θ, (GTP_N), *and* θ_N *are defined as in (III.88), (III.51), (III.86), and (III.83), respectively. Then, the pairs* $((GTP_N),\ \theta_N)$, *in the sequence* $\{((GTP_N), \theta_N)\}_{N \in \mathcal{N}}$ *are consistent approximations for the pair* $((GTP_{cl}), \theta)$.

Proof. Suppose that an infinite sequence $\{\eta_N\}_{N\in\mathcal{N}}$ is such that $\eta_N \in \mathbf{H}^0_N$, for all $N \in \mathcal{N}$, and $\eta_N \to^{\mathcal{N}} \eta$, as $N \to \infty$. Let $\epsilon > 0$. From Lemma III.6 and Lemma III.19 we know that $\nabla f(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0_{cl}$. By the $\mathbf{H}$-continuity of $\nabla f(\cdot)$, there exists an $\bar{N} \in \mathcal{N}$ such that for all $N \geq \bar{N}$, $\frac{K_F}{N} \leq \frac{\epsilon}{2}$ and $\|\nabla f(\eta_N) - \nabla f(\eta)\| \leq \frac{\epsilon}{2}$, where K_F is as in Proposition III.18. Hence, by Proposition III.18

$$\begin{aligned} \|\nabla f_N(\eta_N) - \nabla f(\eta)\|_{H_2} &\leq \|\nabla f_N(\eta_N) - \nabla f(\eta_N)\|_{H_2} + \|\nabla f(\eta_N) - \nabla f(\eta)\|_{H_2} \\ &\leq \frac{K_F}{N} + \frac{\epsilon}{2} \\ &\leq \epsilon, \end{aligned} \tag{III.101}$$

for all $N \geq \bar{N}$, $N \in \mathcal{N}$. Consequently, $\nabla f_N(\eta_N) \to^{\mathcal{N}} \nabla f(\eta)$, as $N \to \infty$, implying that $\theta_N(\eta_N) \to^{\mathcal{N}} \theta(\eta)$, as $N \to \infty$. Theorem III.17, together with the convergence of $\theta_N(\eta_N)$ to $\theta(\eta)$ as $N \to \infty$, satisfies the requirements of Definition III.2 for consistency of approximation. □

Next, we show that the pairs $((GTP^c_N), \theta^c_N)$ in the sequence $\{((GTP^c_N), \theta^c_N)\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((GTP^c), \theta^c)$.

Theorem III.21. *Suppose that Assumptions III.1, III.2, III.3, and III.15 are satisfied, and that* (GTP^c), θ^c, (GTP^c_N), *and* θ^c_N *are defined as in (III.20), (III.52), (III.66), and (III.84), respectively. Then, the pairs* $((GTP^c_N),\ \theta^c_N)$, *in the sequence* $\{((GTP^c_N), \theta^c_N)\}_{N\in\mathcal{N}}$, *are consistent approximations for the pair* $((GTP^c), \theta^c)$.

Proof. The proof that the problems (GTP^c_N) epi-converge to (GTP^c) is the same as the proof of epi-convergence of the problems (GTP_N) to (GTP_{cl}) given in Theorem III.17 above.

From the proof of Theorem III.20 above, we know that $\nabla f_N(\eta_N) \to^{\mathcal{N}} \nabla f(\eta)$, as $N \to \infty$. Then, following the same arguments as in the proof of Theorem

4.3.6 in Polak (1997) we see that given any infinite sequence $\{\eta_N\}_{N\in\mathcal{N}}$, such that $\eta_N \in \mathbf{H}_{c,N}$ for all $N \in \mathcal{N}$, which converges to an $\eta \in \mathbf{H}_c$, $\theta_N^c(\eta_N) \to^{\mathcal{N}} \theta^c(\eta)$, as $N \to \infty$. Theorem III.20, together with the convergence of $\theta_N^c(\eta_N)$ to $\theta^c(\eta)$ as $N \to \infty$, satisfies the requirements of Definition III.2 for consistency of approximation. $\square$

b. *Time- and Space-Discretized Problems*

We next consider the time- and space-discretized problem. As discussed in Section II.A we focus on $A \subset \mathbb{R}^2$, and introduce the space discretization parameter, $M = (M_1, M_2)^T \in \mathbb{N} \times \mathbb{N}$. We also define a generic numerical integration scheme, I_M, for a function, $\Psi : \mathcal{C}^p(A) \to \mathbb{R}$, where $\mathcal{C}^p$ represents the differentiability class.[3] The integration scheme I_M is a mapping from $\mathcal{C}^p(A)$ to $\mathbb{R}$ or to $\mathbb{R}^{n+m}$ for any $M \in \mathbb{N} \times \mathbb{N}$ and $\Psi \in \mathcal{C}^p(A)$ defined by

$$I_M(\Psi) \triangleq \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} W_{ij} \Psi(\alpha_{ij}) \approx \int_{\alpha\in A} \Psi(\alpha) d\alpha, \qquad \text{(III.102)}$$

where W_{ij}, $i = 1, 2, ..., M_1$, $j = 1, 2, ..., M_2$, are weights for the chosen numerical integration scheme, and α_{ij} are the discretization points at which the integrand is evaluated.

We then make use of the integration rule, I_M, to define the approximate objective function $f_{NM} : \mathbf{H}_N \to \mathbb{R}$ for any $\eta \in \mathbf{H}_N$, $N \in \mathcal{N}$, and $M \in \mathbb{N} \times \mathbb{N}$ by

$$f_{NM}(\eta) \triangleq I_M \left(\exp\left[-z_N^\eta(1; \cdot)\right] \phi(\cdot)\right). \qquad \text{(III.103)}$$

We next consider the differentiability of $f_{NM}(\cdot)$.

Proposition III.22. *Suppose that Assumptions III.1, III.2, and III.3 are satisfied, I_M is defined as in (III.102), $N \in \mathcal{N}$, $M \in \mathbb{N} \times \mathbb{N}$, and $f_{NM} : \mathbf{H}_N \to \mathbb{R}$ is defined as in (III.103), then for any $\eta \in \mathbf{H}_N^0$ and $\delta\eta \in H_{\infty,2}$, $f_{NM}(\cdot)$ has a Gateaux differential $Df_{NM}(\eta; \delta\eta) = \langle \nabla f_{NM}(\eta), \delta\eta \rangle_{H_2}$, where*

$$\nabla f_{NM}(\eta)(t) = I_M \left[\nabla_\eta \tilde{f}_N(\eta; \cdot)(t) \phi(\cdot)\right], \forall t \in [0, 1]. \qquad \text{(III.104)}$$

[3] The class $\mathcal{C}^0$ consists of all continuous functions on A. For any positive integer, p, $\mathcal{C}^p$ is the set of all differentiable functions on A whose gradient is in $\mathcal{C}^{p-1}$.

Proof. Let $\delta\eta \in H_{\infty,2}$, and $\eta \in \mathbf{H}_N^0$ be arbitrary. From Lemma III.10(a) we know

$$D\tilde{f}_N(\eta;\alpha;\delta\eta) = \left\langle \nabla_\eta \tilde{f}_N(\eta;\alpha), \delta\eta \right\rangle_{H_2}. \tag{III.105}$$

Then, we have

$$\begin{aligned}
Df_{NM}(\eta;\delta\eta) &= \lim_{\lambda\downarrow 0} \frac{f_{NM}(\eta+\lambda\delta\eta;\alpha) - f_{NM}(\eta;\alpha)}{\lambda} \\
&= \lim_{\lambda\downarrow 0} \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} W_{ij} \left[\frac{\tilde{f}_N(\eta+\lambda\delta\eta;\alpha_{ij}) - \tilde{f}_N(\eta;\alpha_{ij})}{\lambda} \right] \phi(\alpha_{ij}) \\
&= \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} W_{ij} \lim_{\lambda\downarrow 0} \left[\frac{\tilde{f}_N(\eta+\lambda\delta\eta;\alpha_{ij}) - \tilde{f}_N(\eta;\alpha_{ij})}{\lambda} \right] \phi(\alpha_{ij}) \\
&= \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} W_{ij} \left\langle \nabla_\eta \tilde{f}_N(\eta;\alpha_{ij}), \delta\eta \right\rangle_{H_2} \phi(\alpha_{ij}) \\
&= \left\langle \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} W_{ij} \nabla_\eta \tilde{f}_N(\eta;\alpha_{ij}) \phi(\alpha_{ij}), \delta\eta \right\rangle_{H_2}
\end{aligned} \tag{III.106}$$

□

Our next result is related to the Lipschitz $\mathbf{H}_N$-continuity of $\nabla f_{NM}(\eta)$ on bounded subsets of $\mathbf{H}_N^0$.

Lemma III.23. *Suppose that Assumptions III.1, III.2, and III.3 are satisfied, then the gradient $\nabla f_{NM}(\eta)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}_N^0$.*

Proof. The proof follows the same arguments as the proof of Lemma III.12, with integration replaced by I_M. □

In the analysis that follows, it is necessary to quantify the error introduced by the numerical integration scheme in order to complete the convergence proofs. Clearly, the choice of numerical integration scheme determines the relationship between the discretization level and the amount of error introduced by the approximation. In order to conduct our analysis, we make the following assumptions.

Assumption III.24. *We assume that there exist scalars a, b, c, and d such that*

(i) $A = \{(\alpha_1,\alpha_2) \in \mathbb{R}^2 | a \le \alpha_1 \le b, c \le \alpha_2 \le d\}$, where A represents the region of integration in I_M.

(ii) $\alpha_{ij} = (\alpha_1(i), \alpha_2(j))^T$ and an integer M_1 is chosen so that the interval $[a, b]$ is subdivided into $2M_1$ subintervals $\{[\alpha_1(i-1), \alpha_1(i)]\}_{i=1}^{2M_1}$ of equal width $h = (b-a)/2M_1$ by using the equally spaced discretization points $\alpha_1(i) = a + ih$ for $i = 0, 1, 2, ..., 2M_1$. An integer M_2 is chosen so that the interval $[c, d]$ is subdivided into $2M_2$ subintervals $\{[\alpha_2(j-1), \alpha_2(j)]\}_{j=1}^{2M_2}$ of equal width $k = (d-c)/2M_2$ by using the equally spaced discretization points $\alpha_2(j) = c + jh$ for $j = 0, 1, 2, ..., 2M_2$. Composite Simpson's rule is used for numerical integration and for $\Psi \in \mathcal{C}^p(A)$ is given by

$$\begin{aligned} I_M(\Psi) &= \frac{hk}{9}\Bigg(\Psi(a,c) + \Psi(a,d) + \Psi(b,c) + \Psi(b,d) + 4\sum_{j=1}^{M_2}\Psi(a, \alpha_2(2j-1)) \\ &+ 2\sum_{j=1}^{M_2-1}\Psi(a, \alpha_2(2j)) + 4\sum_{j=1}^{M_2}\Psi(b, \alpha_2(2j-1)) + 2\sum_{j=1}^{M_2-1}\Psi(b, \alpha_2(2j)) \\ &+ 4\sum_{i=1}^{M_1}\Psi(\alpha_1(2i-1), c) + 2\sum_{i=1}^{M_1-1}\Psi(\alpha_1(2i), c) + 4\sum_{i=1}^{M_1}\Psi(\alpha_1(2i-1), d) \\ &+ 2\sum_{i=1}^{M_1-1}\Psi(\alpha_1(2i), d) + 16\sum_{j=1}^{M_2}\sum_{i=1}^{M_1}\Psi(\alpha_1(2i-1), \alpha_2(2j-1)) \\ &+ 8\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1}\Psi(\alpha_1(2i-1), \alpha_2(2j)) + 8\sum_{j=1}^{M_2}\sum_{i=1}^{M_1-1}\Psi(\alpha_1(2i), \alpha_2(2j-1)) \\ &+ 4\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1-1}\Psi(\alpha_1(2i), \alpha_2(2j))\Bigg) \end{aligned} \tag{III.107}$$

□

We note that the convergence proofs given below in Proposition III.26 would follow similar arguments if A had higher dimensionality, or a different numerical integration scheme had been utilized. The proofs could also be done if A was assumed to be a shape other than rectangular, but they would be more complicated.

We find it necessary to show that the partial derivatives of $\tilde{f}_N(\eta; \cdot)\phi(\cdot)$ and $\nabla_\eta \tilde{f}_N(\eta; \cdot)\phi(\cdot)$ up to and including the fourth-order are bounded for any $\eta \in \mathbf{H}_N^0$, $\alpha \in A$, and $N \in \mathcal{N}$ in order to complete the proofs of convergence of $f_{NM}(\eta)$ to $f(\eta)$ and $\nabla f_{NM}(\eta)$ to $\nabla f(\eta)$, based on the choice of Composite Simpson's rule as the numerical integration scheme. To facilitate these proofs, we begin by defining some notation. For any $\eta \in \mathbf{H}_N^0$, $\alpha \in A$, $N \in \mathcal{N}$, and $j = 0, 1, ..., N-1$, we define

$$\zeta_1(\alpha) \triangleq \exp\left[-\sum_{j=0}^{N-1}\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L} r^{k,l}\left(x_N^{\eta,k}\left(\frac{j}{N}\right), y^l\left(\frac{j}{N};\alpha\right)\right)\right], \tag{III.108}$$

$$\zeta_2(\alpha) \triangleq \phi(\alpha), \tag{III.109}$$

$$\zeta_3(\alpha) \triangleq p_N^\eta(0;\alpha), \tag{III.110}$$

and

$$\zeta_4(\alpha) \triangleq p_N^\eta\left(\frac{j+1}{N};\alpha\right). \tag{III.111}$$

We note that $\zeta_1(\cdot), \zeta_3(\cdot)$, and $\zeta_4(\cdot)$ depend on η and N. We next show that the partial derivatives of $\zeta_1(\cdot), ..., \zeta_4(\cdot)$ up to and including the fourth order are continuous and bounded for any $\eta \in \mathbf{H}_N^0$, $\alpha \in A$, and $N \in \mathcal{N}$.

Lemma III.25. *Suppose that Assumptions III.1, III.2, and III.3 are satisfied and S is a bounded subset of $\mathbf{H}_N^0$. Then,*

(i) $\zeta_i(\cdot) \in \mathcal{C}^4(A)$, $i = 1, 2, 3, 4$,
  and

(ii) there exists $C < \infty$, such that for all $\eta \in S$, $j = 0, 1, ..., N-1$, $\alpha \in A$, and $N \in \mathcal{N}$

$$\left|\frac{\partial^\mu \zeta_\kappa(\alpha)}{\partial \alpha_i^\mu}\right| \leq C \quad \forall i = 1, 2, \forall \mu = 1, 2, 3, 4, \forall \kappa = 1, 2, 3, 4. \tag{III.112}$$

Proof. It can be seen by repeated applications of the chain and product rules that based on Assumptions III.2(i) and III.3(iii), $\zeta_1(\cdot) \in \mathcal{C}^4(A)$. We also know from Assumption III.1 that $\zeta_2(\cdot) \in \mathcal{C}^4(A)$.

We now consider $\zeta_3(\alpha)$. We know that $p_N^\eta(1;\alpha) \in \mathcal{C}^4(A)$ for the same reasons that $\zeta_1(\cdot) \in \mathcal{C}^4(A)$, because $p_N^\eta(1;\alpha)$ is a column vector of nK zeros and $-\zeta_1(\cdot)$. We also know that $\tilde{h}_x\left(x_N^\eta(\frac{j}{N}), u(\frac{j}{N});\alpha\right)$, $j = 0, 1, ..., N-1$, has $h_x^k\left(x_N^\eta(\frac{j}{N}), u(\frac{j}{N})\right), k = 1, 2, ..., K$, and $\sum_{k=1}^K\sum_{l=1}^L \nabla_x r^{k,l}\left(x_N^\eta(\frac{j}{N}), y^l(\frac{j}{N};\alpha)\right)$ as components. Because there is no α dependence, $h_x^k\left(x_N^\eta(\frac{j}{N}), u(\frac{j}{N})\right)$ and partial derivatives of $h_x^k\left(x_N^\eta(\frac{j}{N}), u(\frac{j}{N})\right)$ with

respect to α are constants for all $k = 1, 2, ..., K$. Based on Assumptions III.2(i) and III.3(iv), $\nabla_x r^{k,l}\left(x_N^\eta(\frac{j}{N}), y^l(\frac{j}{N};\cdot)\right) \in \mathcal{C}^4(A)$. This means that $\tilde{h}_x\left(x_N^\eta(\frac{j}{N}), u(\frac{j}{N});\cdot\right) \in \mathcal{C}^4(A)$. Next, we proceed with an induction argument. Suppose that $p_N^\eta((j+1)/N;\cdot) \in \mathcal{C}^4(A)$, then $p_N^\eta(j/N;\cdot)$, given by

$$p_N^\eta(j/N;\cdot) = p_N^\eta((j+1)/N;\cdot) + \frac{1}{N}\tilde{h}_x(x_N^\eta(j/N), u(j/N);\cdot)^T p_N^\eta((j+1)/N;\cdot), \quad \text{(III.113)}$$

is also $\mathcal{C}^4(A)$. Hence, it then follows by induction that $p_N^\eta(0;\cdot) \in \mathcal{C}^4(A)$, and therefore $\zeta_3(\cdot) \in \mathcal{C}^4(A)$.

By the induction argument above, we know that $p_N^\eta(\frac{j+1}{N};\cdot) \in \mathcal{C}^4(A)$ for all $j = 0, 1, ..., N-2$. Hence, $\zeta_4(\cdot) \in \mathcal{C}^4(A)$, and the proof of part (i) is complete.

We start the proof of part (ii) by looking at $\zeta_1(\alpha)$. It can be seen by repeated use of the product and chain rules that the first through fourth-order partial derivatives of $\zeta_1(\cdot)$ are made up of sums, differences, and products of the expressions

$$\exp\left[-\sum_{j=0}^{N-1}\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L} r^{k,l}\left(x_N^{\eta,k}\left(\frac{j}{N}\right), y^l\left(\frac{j}{N};\alpha\right)\right)\right] \quad \text{(III.114)}$$

and

$$\sum_{j=0}^{N-1}\frac{1}{N}\frac{\partial^\kappa}{\partial\alpha_i^\kappa}\sum_{k=1}^{K}\sum_{l=1}^{L} r^{k,l}\left(x_N^{\eta,k}\left(\frac{j}{N}\right), y^l\left(\frac{j}{N};\alpha\right)\right) \quad \forall i = 1, 2, \forall\kappa = 1, 2, 3, 4. \quad \text{(III.115)}$$

To prove that the first through fourth-order partial derivatives of $\zeta_1(\cdot)$ are bounded, and that the bounds are independent of N and η, we show that expressions (III.114) and (III.115) are bounded with respect to α, and that their bounds are independent of N and η. We first consider (III.114). By Assumption III.3(i), there exists $C_r < \infty$ such that

$$\left|r^{k,l}\left(x_N^{\eta,k}\left(\frac{j}{N}\right), y^l\left(\frac{j}{N};\alpha\right)\right)\right| \leq C_r, \quad \text{(III.116)}$$

for all $\eta \in S$, $k = 1, 2, ..., K$, $l = 1, 2, ..., L$, $\alpha \in A$, $j = 0, 1, ..., N-1$, and $N \in \mathcal{N}$. Then

$$
\begin{aligned}
& \left| \sum_{j=0}^{N-1} \frac{1}{N} \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l} \left(x_N^{\eta,k} \left(\frac{j}{N} \right), y^l \left(\frac{j}{N}; \alpha \right) \right) \right| \\
\leq \quad & \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{K} \sum_{l=1}^{L} \left| r^{k,l} \left(x_N^{\eta,k} \left(\frac{j}{N} \right), y^l \left(\frac{j}{N}; \alpha \right) \right) \right| \\
\leq \quad & \frac{1}{N} \sum_{j=0}^{N-1} KLC_r = KLC_r. \qquad \text{(III.117)}
\end{aligned}
$$

Hence, (III.114) is bounded by $\exp(-KLC_r)$, which is independent of N and η.

Next, we consider (III.115). By Assumptions III.2(ii) and III.3(vi)

$$
\left| \frac{\partial^\kappa}{\partial \alpha_i^\kappa} r^{k,l} \left(x_N^{\eta,k} \left(\frac{j}{N} \right), y^l \left(\frac{j}{N}; \alpha \right) \right) \right| \leq C_{r1} C_y, \qquad \text{(III.118)}
$$

for all $\eta \in S$, $k = 1, 2, ..., K$, $l = 1, 2, ..., L$, $\alpha \in A$, $i = 1, 2$, $j = 0, 1, ..., N-1$, $\kappa = 1, 2, 3, 4$, and $N \in \mathcal{N}$, with C_{r1} and C_y as in Assumptions III.2(ii) and III.3(vi), respectively. Then, for all $i = 1, 2$, and $\kappa = 1, 2, 3, 4$

$$
\left| \sum_{j=0}^{N-1} \frac{1}{N} \frac{\partial^\kappa}{\partial \alpha_i^\kappa} \sum_{k=1}^{K} \sum_{l=1}^{L} r^{k,l} \left(x_N^{\eta,k} \left(\frac{j}{N} \right), y^l \left(\frac{j}{N}; \alpha \right) \right) \right| \qquad \text{(III.119)}
$$

$$
\begin{aligned}
\leq \quad & \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{K} \sum_{l=1}^{L} \left| \frac{\partial^\kappa}{\partial \alpha_i^\kappa} r^{k,l} \left(x_N^{\eta,k} \left(\frac{j}{N} \right), y^l \left(\frac{j}{N}; \alpha \right) \right) \right| \\
\leq \quad & \frac{1}{N} \sum_{j=0}^{N-1} KLC_{r1}C_y = KLC_{r1}C_y. \qquad \text{(III.120)}
\end{aligned}
$$

Therefore, (III.115) is bounded by $KLC_{r1}C_y$, which is independent of N and η. Because (III.114) and (III.115) are bounded, and their bounds are independent of N and η, the first- through fourth-order partial derivatives of $\zeta_1(\alpha)$ with respect to α are bounded, and the bounds are independent of N and η.

Now, we consider $\zeta_2(\alpha)$. From Assumption III.1 and the compactness of A, we know that the first- through fourth-order partial derivatives of $\zeta_2(\alpha)$ are bounded for any $\alpha \in A$.

We next consider $\zeta_3(\alpha)$. We know that the first- through fourth-order partial derivatives of $p_N^\eta(1;\alpha)$ with respect to α, are bounded for the same reasons that the first- through fourth-order partial derivatives of $\zeta_1(\alpha)$ are bounded. The first- through fourth-order partial derivatives of $h_x^k\left(x_N^\eta(\cdot), u(\cdot)\right)$, $k = 1, 2, ..., K$, with respect to α are bounded because they are all equal to zero. The first- through fourth-order partial derivatives of $\nabla_x r^{k,l}\left(x_N^\eta(t), y^l(t;\alpha)\right)$ with respect to α are bounded for any $t \in [0,1]$ by Assumptions III.2(ii) and III.3(vi), and the bounds are independent of t and N. Then, because $\tilde{h}_x(x(t), u(t);\alpha)$ is earlier defined by

$$\tilde{h}_x(x(t), u(t);\alpha) \triangleq \begin{pmatrix} h_x^1(x(t), u(t))^T \\ \vdots \\ h_x^K(x(t), u(t))^T \\ \sum_{k=1}^K \sum_{l=1}^L \nabla_x r^{k,l}(x^k(t), y^l(t;\alpha))^T \end{pmatrix}, \tag{III.121}$$

there exists $K'_{\tilde{h}_x} < \infty$ such that

$$\left\| \frac{\partial^\kappa}{\partial \alpha_i^\kappa} \tilde{h}_x\left(x_N^\eta(j/N), u(j/N);\alpha\right) \right\| \leq K'_{\tilde{h}_x}, \tag{III.122}$$

for all $\eta \in S$, $\alpha \in A$, $i = 1, 2$, $j = 0, 1, ..., N-1$, $\kappa = 1, 2, 3, 4$, and $N \in \mathcal{N}$. From Assumptions III.3(ii), (iii), and (v), we know that $\tilde{h}_x\left(x_N^\eta(j/N), u(j/N);\alpha\right)$ is continuous with respect to α for all $\eta \in S$, on A. Since $\tilde{h}_x\left(x_N^\eta(j/N), u(j/N);\alpha\right)$ is continuous with respect to α, and S is a bounded subset of $\mathbf{H}_N^0$, there exists $K_{\tilde{h}_x} < \infty$ such that

$$\left\| \tilde{h}_x\left(x_N^\eta(j/N), u(j/N);\alpha\right) \right\| \leq K_{\tilde{h}_x}, \tag{III.123}$$

for all $\eta \in S$, $\alpha \in A$, $j = 0, 1, ..., N-1$, and $N \in \mathcal{N}$. Because $z_N^\eta(1;\alpha) \geq 0$ for all $\eta \in S$ and $\alpha \in A$, there exists $C_1 < \infty$ such that $\|p_N^\eta(1;\alpha)\| \leq C_1$ for all $\eta \in S$, $\alpha \in A$, and $N \in \mathcal{N}$. To show that $p_N^\eta(j/N;\alpha)$ is bounded for all $\eta \in S$, $\alpha \in A$, and $j = 0, 1, ..., N-1$, we proceed with an induction argument. Suppose that there exists $C_{j+1} < \infty$, such that $\|p_N^\eta((j+1)/N;\alpha)\| \leq C_{j+1}$, for all $\eta \in S$, $\alpha \in A$, and $N \in \mathcal{N}$.

Then, based on (III.113)

$$\begin{aligned} \|p_N^\eta(j/N;\alpha)\| &\leq \|p_N^\eta((j+1)/N;\alpha)\| \\ &+ \frac{1}{N}\left\|\tilde{h}_x(x_N^\eta(j/N),u(j/N);\alpha)^T p_N^\eta((j+1)/N;\alpha)\right\| \\ &\leq C_{j+1} + \frac{1}{N}K_{\tilde{h}_x}C_{j+1} = C_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right) \end{aligned} \quad \text{(III.124)}$$

By doing another step in the backward recursion, we find that

$$\begin{aligned} \|p_N^\eta((j-1)/N;\alpha)\| &\leq C_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right) + \frac{K_{\tilde{h}_x}}{N}C_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right) \\ &= C_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right)^2 \\ &\leq C_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right)^N \end{aligned} \quad \text{(III.125)}$$

There exists $\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$, $\left(1+\frac{K_{\tilde{h}_x}}{N}\right)^N \leq 2e^{K_{\tilde{h}_x}}$. Then, for any $N \geq \bar{N}$,

$$\|p_N^\eta((j-1)/N;\alpha)\| \leq 2C_{j+1}e^{K_{\tilde{h}_x}}. \quad \text{(III.126)}$$

For values of N smaller than $\bar{N}$, we define

$$\delta_i \triangleq \left(1+\frac{K_{\tilde{h}_x}}{\bar{N}-i}\right)^{\bar{N}-i}, \; i=1,2,...,\bar{N}-1, \quad \text{(III.127)}$$

and

$$K_\delta \triangleq \max\{\max_{i=1,2,\ldots,\bar{N}-1}\{\delta_i\},2\}. \quad \text{(III.128)}$$

Then,

$$\|p_N^\eta((j-1)/N;\alpha)\| \leq K_\delta C_{j+1}e^{K_{\tilde{h}_x}}. \quad \text{(III.129)}$$

Hence, it follows by induction that $p_N^\eta(j/N;\alpha)$ is bounded for all $\eta \in S$, $\alpha \in A$, $j = 0,1,...,N-1$, and the bound is independent of N and η.

We now proceed with another induction argument to show that the first- through fourth-order partial derivatives of $p_N^\eta(j/N;\alpha)$ with respect to α are bounded. Suppose the first- through fourth-order partial derivatives of $p_N^\eta((j+$

$1)/N;\alpha)$ with respect to α are bounded and that the bound, $C'_{j+1} < \infty$, is independent of N and η. Then, based on (III.113), for all $\eta \in S$, $\alpha \in A$, $i = 1,2$, $j = 0,1,...,N-1$, $\kappa = 1,2,3,4$, and $N \in \mathcal{N}$

$$\left\|\frac{\partial^\kappa}{\partial\alpha_i^\kappa} p_N^\eta(j/N;\alpha)\right\| \leq \left\|\frac{\partial^\kappa}{\partial\alpha_i^\kappa} p_N^\eta((j+1)/N;\alpha)\right\|$$

$$+\frac{1}{N}\left\|\frac{\partial^\kappa}{\partial\alpha_i^\kappa}\left(\tilde{h}_x(x_N^\eta(j/N),u(j/N);\alpha)^T p_N^\eta((j+1)/N;\alpha)\right)\right\|$$

$$\begin{aligned}
&\leq C'_{j+1} + \frac{1}{N}\left\|\tilde{h}_x(x_N^\eta(j/N),u(j/N);\alpha)^T \frac{\partial^\kappa}{\partial\alpha_i^\kappa} p_N^\eta((j+1)/N;\alpha)\right\| \\
&+ \frac{1}{N}\left\|\left(\frac{\partial^\kappa}{\partial\alpha_i^\kappa}\tilde{h}_x(x_N^\eta(j/N),u(j/N);\alpha)^T\right) p_N^\eta((j+1)/N;\alpha)\right\| \\
&\leq C'_{j+1} + \frac{K_{\tilde{h}_x}C'_{j+1} + C_{j+1}K'_{\tilde{h}_x}}{N} \\
&\leq C'_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right) + \frac{K'_{\tilde{h}_x}C_{j+1}}{N} \qquad \text{(III.130)}
\end{aligned}$$

By doing another step in the backward recursion, we find

$$\begin{aligned}
\left\|\frac{\partial^\kappa}{\partial\alpha_i^\kappa} p_N^\eta((j-1)/N;\alpha)\right\| &\leq C'_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right)^2 + \frac{K'_{\tilde{h}_x}C_{j+1}}{N} + \frac{K_{\tilde{h}_x}K'_{\tilde{h}_x}C_{j+1}}{N^2} + \frac{C_jK'_{\tilde{h}_x}}{N} \\
&\leq C'_{j+1}\left(1+\frac{K_{\tilde{h}_x}}{N}\right)^N + \frac{K'_{\tilde{h}_x}C_{j+1}}{N} + \frac{K_{\tilde{h}_x}K'_{\tilde{h}_x}C_{j+1}}{N^2} + \frac{C_jK'_{\tilde{h}_x}}{N} \\
&\leq K_\delta C'_{j+1}e^{K_{\tilde{h}_x}} + K_C, \qquad \text{(III.131)}
\end{aligned}$$

where $K_C < \infty$ is such that

$$K_C \geq \frac{K'_{\tilde{h}_x}C_{j+1}}{N} + \frac{K_{\tilde{h}_x}K'_{\tilde{h}_x}C_{j+1}}{N^2} + \frac{C_jK'_{\tilde{h}_x}}{N} \qquad \text{(III.132)}$$

for any $N \in \mathcal{N}$. Hence, it follows by induction that the first- through fourth-order partial derivatives of $p_N^\eta(0;\cdot)$ with respect to α are bounded, and therefore the first- through fourth-order partial derivatives of $\zeta_3(\cdot)$ with respect to α are bounded. Furthermore, these bounds are independent of N and η.

By the induction argument above, the first- through fourth-order partial derivatives of $p_N^\eta(\frac{j+1}{N};\cdot)$ with respect to α are bounded for all $j = 0,1,...,N-2$

and the bounds are independent of N and η. This means that the first- through fourth-order partial derivatives of $\zeta_4(\cdot)$ with respect to α are bounded, and the proof of part (ii) is complete. □

In order to prove epi-convergence and consistency of approximation, we need the following proposition.

Proposition III.26. *Suppose that Assumptions III.1, III.2, III.3, and III.24 are satisfied. Then for every bounded subset $S \subset \mathbf{H}^0$, there exist constants $K_{I1}, K_{I2} < \infty$ such that, for any $N \in \mathcal{N}$, $M \in \mathbb{N}_3 \times \mathbb{N}_3$, where $\mathbb{N}_3 \triangleq \{m \in 2\mathbb{N}+1 | m \geq 3\}$, and $\eta \in S \cap H_N$,*

(i)

$$|f(\eta) - f_{NM}(\eta)| \leq \frac{K_S}{N} + \frac{K_{I1}}{(M_1-1)^4} + \frac{K_{I2}}{(M_2-1)^4} \tag{III.133}$$

and

(ii)

$$\|\nabla f(\eta) - \nabla f_{NM}(\eta)\|_{H_2} \leq \frac{K_F}{N} + \frac{K_{I1}}{(M_1-1)^4} + \frac{K_{I2}}{(M_2-1)^4}, \tag{III.134}$$

and K_S and K_F are defined as in (III.92) and Proposition III.18, respectively.

Proof. We know that

$$|f(\eta) - f_{NM}(\eta)| \leq |f(\eta) - f_N(\eta)| + |f_N(\eta) - f_{NM}(\eta)|, \tag{III.135}$$

and

$$\|\nabla f(\eta) - \nabla f_{NM}(\eta)\|_{H_2} \leq \|\nabla f(\eta) - \nabla f_N(\eta)\|_{H_2} + \|\nabla f_N(\eta) - \nabla f_{NM}(\eta)\|_{H_2}. \tag{III.136}$$

Based on Proposition III.16 and Proposition III.18, respectively, $|f(\eta) - f_N(\eta)| \leq \frac{K_S}{N}$ and $\|\nabla f(\eta) - \nabla f_N(\eta)\|_{H_2} \leq \frac{K_F}{N}$, for any $\eta \in S \cap \mathbf{H}_N$. In order to deal with $|f_N(\eta) - f_{NM}(\eta)|$ and $\|\nabla f_N(\eta) - \nabla f_{NM}(\eta)\|_{H_2}$, we define for notational simplicity

$$g_1(\alpha) \triangleq \zeta_1(\alpha)\zeta_2(\alpha), \tag{III.137}$$

and

$$g_2(\alpha) \triangleq \nabla \tilde{f}_N(\eta; \alpha)\phi(\alpha), \tag{III.138}$$

where by Lemma III.10 and the definitions given in (III.109), (III.110), and (III.111) we can write out the components of $g_2(\alpha)$ as

$$g_{2\xi}(\alpha) = \nabla_\xi \tilde{f}_N(\eta;\alpha)\phi(\alpha) = p_N^\eta(0;\alpha)\phi(\alpha) = \zeta_3(\alpha)\zeta_2(\alpha), \tag{III.139}$$

and for $t \in [0,1]$

$$\begin{aligned} g_{2ut}(\alpha) &= \nabla_u \tilde{f}_N(\eta;\alpha)(t)\phi(\alpha) = \left[\sum_{j=0}^{N-1} \gamma_N^\eta\left(\frac{j}{N};\alpha\right)\pi_{N,j}(t)\right]\phi(\alpha) \\ &= \sum_{j=0}^{N-1} \begin{pmatrix} h_u^1\left(x_N^\eta\left(\frac{j}{N}\right), u\left(\frac{j}{N}\right)\right) \\ \vdots \\ h_u^K\left(x_N^\eta\left(\frac{j}{N}\right), u\left(\frac{j}{N}\right)\right) \\ 0 \end{pmatrix}^T \pi_{N,j}(t)\zeta_4(\alpha)\zeta_2(\alpha). \end{aligned} \tag{III.140}$$

Then, by Lemma III.25(i), $g_1(\cdot) \in \mathcal{C}^4(A)$ and $g_2(\cdot) \in \mathcal{C}^4(A)$. Again, for notational simplicity, we define

$$E_1 \triangleq \left| I_M\left(g_1(\cdot)\right) - \int_{\alpha\in A} g_1(\alpha)d\alpha \right|, \tag{III.141}$$

$$E_{2\xi i} \triangleq \left| I_M(g_{2\xi i}(\cdot)) - \int_{\alpha\in A} g_{2\xi i}(\alpha)d\alpha \right|, \tag{III.142}$$

and

$$E_{2utj} \triangleq \left| I_M(g_{2utj}(\cdot)) - \int_{\alpha\in A} g_{2utj}(\alpha)d\alpha \right|, \tag{III.143}$$

where $g_{2\xi i}(\alpha)$ is the i^{th} component of $g_{2\xi}(\alpha)$ and $g_{2utj}(\alpha)$ is the j^{th} component of $g_{2ut}(\alpha)$. From Lemma III.25(ii) for $\rho = 1, 2\xi i, 2utj$, $i = 1, 2, ..., nK+1$, $t \in [0,1]$, and $j = 1, 2, ..., m$, there exist constants $C_1, C_2 < \infty$ that have no dependence on η, α, or N such that

$$\max_{\alpha\in A} \left| \frac{\partial^4 g_\rho(\alpha)}{\partial \alpha_1^4} \right| \le C_1 \tag{III.144}$$

and

$$\max_{\alpha\in A} \left| \frac{\partial^4 g_\rho(\alpha)}{\partial \alpha_2^4} \right| \le C_2. \tag{III.145}$$

Then, under Assumption III.24, when using Composite Simpson's rule to approximate the integral of $g_1(\cdot)$, E_1 is bounded by (see, for example, pages 127–128 in Faires & Burden, 1993)

$$\begin{aligned} E_1 &\leq \frac{(d-c)(b-a)}{180}\left[\frac{(b-a)^4}{(M_1-1)^4}\max_{\alpha\in A}\left|\frac{\partial^4 g_1(\alpha)}{\partial\alpha_1^4}\right| + \frac{(d-c)^4}{(M_2-1)^4}\max_{\alpha\in A}\left|\frac{\partial^4 g_1(\alpha)}{\partial\alpha_2^4}\right|\right] \\ &\leq \frac{(d-c)(b-a)}{180}\left[\frac{(b-a)^4}{(M_1-1)^4}C_1 + \frac{(d-c)^4}{(M_2-1)^4}C_2\right] \leq \frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4}, \end{aligned}$$

where

$$\tilde{K}_{I1} = \frac{C_1(d-c)(b-a)^5}{180}, \tag{III.146}$$

and

$$\tilde{K}_{I2} = \frac{C_2(b-a)(d-c)^5}{180}. \tag{III.147}$$

Under Assumption III.24, when using Composite Simpson's rule to approximate the integral of $g_{2\xi i}(\cdot)$, $E_{2\xi i}$ is bounded for $i = 1, 2, ..., nK+1$ by

$$\begin{aligned} E_{2\xi i} &\leq \frac{(d-c)(b-a)}{180}\left[\frac{(b-a)^4}{(M_1-1)^4}\max_{\alpha\in A}\left|\frac{\partial^4 g_{2\xi i}(\alpha)}{\partial\alpha_1^4}\right| + \frac{(d-c)^4}{(M_2-1)^4}\max_{\alpha\in A}\left|\frac{\partial^4 g_{2\xi i}(\alpha)}{\partial\alpha_2^4}\right|\right] \\ &\leq \frac{(d-c)(b-a)}{180}\left[\frac{(b-a)^4}{(M_1-1)^4}C_1 + \frac{(d-c)^4}{(M_2-1)^4}C_2\right] \leq \frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4}, \end{aligned}$$

where $\tilde{K}_{I1}$ and $\tilde{K}_{I2}$ are as above. Similarly, under Assumption III.24, when using Composite Simpson's rule to approximate the integral of $g_{2utj}(\cdot)$, E_{2utj} is bounded for $t \in [0,1]$ and $j = 1, 2, ..., m$ by

$$\begin{aligned} E_{2utj} &\leq \frac{(d-c)(b-a)}{180}\left[\frac{(b-a)^4}{(M_1-1)^4}\max_{\alpha\in A}\left|\frac{\partial^4 g_{2ut}(\alpha)}{\partial\alpha_1^4}\right| + \frac{(d-c)^4}{(M_2-1)^4}\max_{\alpha\in A}\left|\frac{\partial^4 g_{2ut}(\alpha)}{\partial\alpha_2^4}\right|\right] \\ &\leq \frac{(d-c)(b-a)}{180}\left[\frac{(b-a)^4}{(M_1-1)^4}C_1 + \frac{(d-c)^4}{(M_2-1)^4}C_2\right] \leq \frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4}, \end{aligned}$$

where, again, $\tilde{K}_{I1}$ and $\tilde{K}_{I2}$ are as above. Then, we have

$$\begin{aligned}
& \|\nabla f_N(\eta) - \nabla f_{NM}(\eta)\|_{H_2}^2 \\
= & \|\nabla_\xi f_N(\eta) - \nabla_\xi f_{NM}(\eta)\|^2 + \int_0^1 \|\nabla_u f_N(\eta)(t) - \nabla_u f_{NM}(\eta)(t)\|^2 dt \\
\leq & \sum_{i=1}^{nK+1} \left(\frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4} \right)^2 + \sum_{j=1}^{m} \left(\frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4} \right)^2 \\
= & (nK+1+m) \left(\frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4} \right)^2 . \qquad \text{(III.148)}
\end{aligned}$$

We let $K_{I1} = \sqrt{nK+1+m}\tilde{K}_{I1}$ and $K_{I2} = \sqrt{nK+1+m}\tilde{K}_{I2}$, and (III.134) follows from (III.136). Finally,

$$\left| I_M\left(g_1(\cdot)\right) - \int_{\alpha \in A} g_1(\alpha) d\alpha \right| \leq \frac{\tilde{K}_{I1}}{(M_1-1)^4} + \frac{\tilde{K}_{I2}}{(M_2-1)^4} \leq \frac{K_{I1}}{(M_1-1)^4} + \frac{K_{I2}}{(M_2-1)^4}, \qquad \text{(III.149)}$$

and (III.133) follows from (III.135), which completes the proof. □

For any $N \in \mathcal{N}$, and $M \in \mathbb{N} \times \mathbb{N}$, we define the following approximating problems

$$(GTP_{NM}) \quad \min_{\eta \in \mathbf{H}_N^0} \quad f_{NM}(\eta), \qquad \text{(III.150)}$$

and

$$(GTP_{NM}^c) \quad \min_{\eta \in \mathbf{H}_{c,N}} \quad f_{NM}(\eta). \qquad \text{(III.151)}$$

For any $N \in \mathcal{N}$, and $M \in \mathbb{N} \times \mathbb{N}$, we define nonpositive optimality functions $\theta_{NM} : \mathbf{H}_N^0 \to \mathbb{R}$ and $\theta_{NM}^c : \mathbf{H}_{c,N} \to \mathbb{R}$ by

$$\theta_{NM}(\eta) \triangleq -\frac{1}{2}\|\nabla f_{NM}(\eta)\|_{H_2}^2, \qquad \text{(III.152)}$$

and

$$\theta_{NM}^c(\eta) \triangleq \min_{\eta' \in \mathbf{H}_{c,N}} \langle \nabla f_{NM}(\eta), \eta' - \eta \rangle_{H_2} + \frac{1}{2}\left\|\eta' - \eta\right\|_{H_2}^2, \qquad \text{(III.153)}$$

which characterize stationary points of (GTP_{NM}) and (GTP_{NM}^c), respectively.

Proposition III.27. *Suppose that Assumptions III.1, III.2, III.3, and III.24 are satisfied.*

(a) $\theta_{NM}(\cdot)$ and $\theta^c_{NM}(\cdot)$ are $\mathbf{H}^0_N$-continuous functions.

(b) If $\hat{\eta} \in \mathbf{H}^0_N$ is a local minimizer of (GTP_{NM}), then $\theta_{NM}(\hat{\eta}) = 0$.

(c) If $\hat{\eta} \in \mathbf{H}_{c,N}$ is a local minimizer of (GTP^c_{NM}), then $\theta^c_{NM}(\hat{\eta}) = 0$.

Proof. The proof follows the same arguments as the proof of Proposition 1.1.6 in Polak (1997), with the norms and inner products replaced by their H_2 equivalents. □

The proofs of convergence that follow require that we have only one discretization parameter, therefore for $i = 1, 2$, we define $M_i : \mathcal{N} \to \mathbb{N}$ and make the following assumption.

Assumption III.28. *We assume for $i = 1, 2$ that $M_i(N) \to \infty$, as $N \to \infty$.* □

It is not possible to establish epi-convergence of $(GTP_{NM(N)})$ to (GTP), but it is possible to establish epi-convergence of $(GTP_{NM(N)})$ to (GTP_{cl}). We now show that the family $\{((GTP_{NM(N)}), \theta_{NM(N)})\}_{N\in\mathcal{N}}$ is a sequence of consistent approximations for $((GTP_{cl}), \theta)$ by showing that the conditions of Definition III.2 are satisfied.

Theorem III.29. *Suppose that Assumptions III.1, III.2, III.3, III.15, and III.28 are satisfied, (GTP_{cl}), θ, $(GTP_{NM(N)})$, and $\theta_{NM(N)}$ are defined as in (III.88), (III.51), (III.150), and (III.152), respectively. Then the pairs $((GTP_{NM(N)}),\ \theta_{NM(N)})$, in the sequence $\{((GTP_{NM(N)}), \theta_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((GTP_{cl}), \theta)$.*

Proof. We first show that $(GTP_{NM(N)})$ epi-converges to (GTP_{cl}), as $N \to \infty$, by showing that the conditions in Proposition III.14 are satisfied. We begin by showing that part (a) of Proposition III.14 is satisfied. Let $\eta \in \mathbf{H}^0_{cl}$ be arbitrary. Then from Proposition III.8 there exists a sequence $\{\eta_N\}_{N\in\mathcal{N}}$ such that $\eta_N \in \mathbf{H}^0_N$, for all $N \in \mathcal{N}$, and $\eta_N \to^{\mathcal{N}} \eta$ as $N \to \infty$. Let $\epsilon > 0$. By Assumption III.28 and the $\mathbf{H}$-continuity of $f(\cdot)$, there exists an $\bar{N} \in \mathcal{N}$ such that for all $N \geq \bar{N}$, $\frac{K_{I1}}{(M_1(N)-1)^4} + \frac{K_{I2}}{(M_2(N)-1)^4} \leq \frac{\epsilon}{2}$, $\frac{K_S}{N} \leq \frac{\epsilon}{4}$, and $|f(\eta_N) - f(\eta)| \leq \frac{\epsilon}{4}$, where K_{I1} and K_{I2} are as in Proposition III.26, and

K_S is as in (III.92). Hence, by Proposition III.26(i),

$$\begin{aligned}\left|f_{NM(N)}(\eta_N) - f(\eta)\right| &\le \left|f_{NM(N)}(\eta_N) - f(\eta_N)\right| + \left|f(\eta_N) - f(\eta)\right| \\ &\le \frac{K_S}{N} + \frac{K_{I1}}{(M_1(N)-1)^4} + \frac{K_{I2}}{(M_2(N)-1)^4} + \frac{\epsilon}{4} \\ &\le \epsilon \qquad \text{(III.154)}\end{aligned}$$

for all $N \ge \bar{N}$, $N \in \mathcal{N}$. Consequently, $f_{NM(N)}(\eta_N) \to^{\mathcal{N}} f(\eta)$, as $N \to \infty$, which completes the proof of condition (a) of Proposition III.14.

In order to show that condition (b) of Proposition III.14 is satisfied, suppose that a sequence $\{\eta_N\}_{N\in\mathcal{N}}$ is such that $\eta_N \in \mathbf{H}_N^0$ for all $N \in \mathcal{N}$, and $\eta_N \to^{\mathcal{N}} \eta$, as $N \to \infty$. Based on the construction of $\mathbf{H}_N^0$, we must have that $\eta \in \mathbf{H}_{cl}^0$. It again follows from the $\mathbf{H}$-continuity of $f(\cdot)$ and Proposition III.26(i) that $f_{NM(N)}(\eta_N) \to^{\mathcal{N}} f(\eta)$ as $N \to \infty$, which satisfies condition (b) of Proposition III.14. This proves that $(GTP_{NM(N)})$ epi-converges to (GTP_{cl}).

For the second part of the proof, we show the convergence of the optimality functions. Suppose that an infinite sequence $\{\eta_N\}_{N\in\mathcal{N}}$ is such that $\eta_N \in \mathbf{H}_N^0$, for all $N \in \mathcal{N}$, and $\eta_N \to^{\mathcal{N}} \eta$ as $N \to \infty$. Let $\epsilon > 0$. From Lemma III.6 and Lemma III.19 we know that $\nabla f(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}_{cl}^0$. By Assumption III.28 and the $\mathbf{H}$-continuity of $\nabla f(\cdot)$, there exists an $\bar{N} \in \mathcal{N}$ such that for all $N \ge \bar{N}$, $\frac{K_{I1}}{(M_1(N)-1)^4} + \frac{K_{I2}}{(M_2(N)-1)^4} \le \frac{\epsilon}{2}$, $\frac{K_F}{N} \le \frac{\epsilon}{4}$ and $\|\nabla f(\eta_N) - \nabla f(\eta)\|_{H_2} \le \frac{\epsilon}{4}$, where K_{I1} and K_{I2} are as in Proposition III.26, and K_F is as in Proposition III.18. Hence, by Proposition III.26(ii),

$$\begin{aligned}&\|\nabla f_{NM(N)}(\eta_N) - \nabla f(\eta)\|_{H_2} \\ \le\ &\|\nabla f_{NM(N)}(\eta_N) - \nabla f(\eta_N)\|_{H_2} + \|\nabla f(\eta_N) - \nabla f(\eta)\|_{H_2} \\ \le\ &\frac{K_F}{N} + \frac{K_{I1}}{(M_1(N)-1)^4} + \frac{K_{I2}}{(M_2(N)-1)^4} + \frac{\epsilon}{4} \\ \le\ &\epsilon, \qquad \text{(III.155)}\end{aligned}$$

for all $N \ge \bar{N}$, $N \in \mathcal{N}$. Consequently, $\nabla f_{NM(N)}(\eta_N) \to^{\mathcal{N}} \nabla f(\eta)$ as $N \to \infty$, and therefore $\theta_{NM(N)}(\eta_N) \to^{\mathcal{N}} \theta(\eta)$ as $N \to \infty$. The epi-convergence of $(GTP_{NM(N)})$ to

(GTP_{cl}) as $N \to \infty$, together with the convergence of $\theta_{NM(N)}(\eta_N)$ to $\theta(\eta)$ as $N \to \infty$, satisfy the requirements of Definition III.2 for consistency of approximation, which completes the proof. $\square$

We now show that the pairs $((GTP^c_{NM(N)}), \theta^c_{NM(N)})$ in the sequence $\{((GTP^c_{NM(N)}), \theta^c_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((GTP^c), \theta^c)$.

Theorem III.30. *Suppose that Assumptions III.2, III.3, III.15, and III.28 are satisfied, (GTP^c), θ^c, $(GTP^c_{NM(N)})$, and $\theta^c_{NM(N)}$ are defined as in (III.20), (III.52), (III.151), and (III.153), respectively. Then the pairs $((GTP^c_{NM(N)}),\ \theta^c_{NM(N)})$, in the sequence $\{((GTP^c_{NM(N)}), \theta^c_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((GTP^c), \theta^c)$.*

Proof. The proof that the problems $(GTP^c_{NM(N)})$ epi-converge to (GTP^c) is the same as the proof that the problems $(GTP_{NM(N)})$ epi-converge to (GTP_{cl}) given in Theorem III.29 above.

From the proof of Theorem III.29 above, we know that $\nabla f_{NM(N)}(\eta_N) \to^{\mathcal{N}} \nabla f(\eta)$, as $N \to \infty$. Then, following the same arguments as in the proof of Theorem 4.3.6 in Polak (1997) we see that given any infinite sequence $\{\eta_N\}_{N\in\mathcal{N}}$, such that $\eta_N \in \mathbf{H}_{c,N}$ for all $N \in \mathcal{N}$, which converges to an $\eta \in \mathbf{H}_c$, $\theta^c_{NM(N)}(\eta_N) \to^{\mathcal{N}} \theta^c(\eta)$, as $N \to \infty$. $\square$

C. INDEPENDENT TARGETS

We now consider the case of independent targets, where we proceed in a manner similar to that used in Section III.B for the case of dependent targets. While our assumptions and approach are similar to those used in Section III.B, the "information state" for the case of independent targets is different than it was for the case of dependent targets. The difference in the "information state" is explained in detail in Section III.C.1.

This section provides discretization schemes that lead to implementable algorithms for the problems (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$, which were defined in Chapter II. The definitions of these problems in Chapter II were incomplete be-

cause they did not include definitions for the spaces of allowable controls. This section begins by defining an "information state" which we use in conjunction with the spaces from Section III.A to complete the definitions of (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$. Next, we state our assumptions and define optimality conditions for the generalized optimal control problems. Then, we develop consistent approximations for the time-discretized search problems. Finally, we show that the time- and space-discretized search problems are consistent approximations for the original, continuous time-and-space search problems.

1. Information State and Optimal Control Problems

Under the assumption that the random variables that the target motion is conditioned upon are independent across targets, we use (II.9) to define the function $f^l : \mathbf{H} \to \mathbb{R}$ for any $\eta \in \mathbf{H}$ by

$$f^l(\eta) \triangleq \int_{\alpha^l \in A} \exp\left(- \int_0^1 \sum_{k=1}^{K} r^{k,l}(x^{\eta,k}(t), y^l(t;\alpha^l))dt \right) \phi^l(\alpha^l)d\alpha^l. \tag{III.156}$$

As before, the superscript l is used to denote the l^{th} attacker and $l = 1, 2, ..., L$. Again, to simplify the notation in (III.156) and facilitate the development that follows, we find it useful to define a parametric "information state" denoted by $z^{\eta,l}(t;\alpha^l)$. For any $\alpha^l \in A$, $l = 1, 2, ..., L$, $t \in [0, 1]$, and set of searcher trajectories, $z^{\eta,l}(t;\alpha^l)$ represents the cumulative detection rate given those searcher trajectories and α^l, and is given by

$$z^{\eta,l}(t;\alpha^l) \triangleq \int_0^t \sum_{k=1}^{K} r^{k,l}\left(x^{\eta,k}(s), y^l(s;\alpha^l)\right) ds, \tag{III.157}$$

or equivalently by the differential equation

$$\dot{z}^{\eta,l}(s;\alpha^l) = \sum_{k=1}^{K} r^{k,l}\left(x^{\eta,k}(s), y^l(s;\alpha^l)\right) \ \forall \ s \in [0, t], \tag{III.158}$$

with $z^{\eta,l}(0;\alpha^l) = 0$. It is important to note that the "information state" in (III.157) differs from the "information state" in (III.12) for the dependent target case. In the

dependent target case, the "information state" represented the cumulative detection rate for all of the searchers looking for all of the targets. For the independent target case, the "information state" represents the cumulative detection rate for all of the searchers looking for the l^{th} target. Using this notation, for any $\eta \in \mathbf{H}$, (III.156) simplifies to

$$f^l(\eta) \triangleq \int_{\alpha^l \in A} \exp\left(-z^{\eta,l}(1;\alpha^l)\right) \phi^l(\alpha^l) d\alpha^l. \tag{III.159}$$

Again, it is useful to simplify the notation in (III.159) even further. To this end, for any $\alpha^l \in A$, $l = 1, 2, ..., L$, we also define the function $\tilde{f}^l(\cdot;\alpha^l) : \mathbf{H} \to \mathbb{R}$ by

$$\tilde{f}^l(\eta;\alpha^l) \triangleq F\left(\tilde{\xi}, \tilde{x}^{\eta,l}(t;\alpha^l)\right), \tag{III.160}$$

where $F(\cdot;\cdot)$ is defined as in (III.15), and $\tilde{x}^{\eta,l}(t;\alpha^l)$ is an augmented state defined by

$$\tilde{x}^{\eta,l}(t;\alpha^l) \triangleq \begin{pmatrix} x^\eta(t) \\ z^{\eta,l}(t;\alpha^l) \end{pmatrix} \in \mathbb{R}^{nK+1}. \tag{III.161}$$

Using this notation, for any $\eta \in \mathbf{H}$, (III.159) simplifies to

$$f^l(\eta) \triangleq \int_{\alpha^l \in A} \tilde{f}^l(\eta;\alpha^l)\phi^l(\alpha^l) d\alpha^l. \tag{III.162}$$

To formulate the problem of maximizing the expected number of targets detected during $[0,1]$, we define the objective function $\psi^e : \mathbf{H} \to \mathbb{R}$ for any $\eta \in \mathbf{H}$ by

$$\psi^e(\eta) \triangleq \sum_{l=1}^{L} f^l(\eta). \tag{III.163}$$

Then, we consider the problem

$$(ITP^e) \quad \min_{\eta \in \mathbf{H}^0} \quad \psi^e(\eta). \tag{III.164}$$

We also consider the problem

$$(ITP^{c,e}) \quad \min_{\eta \in \mathbf{H}_c} \quad \psi^e(\eta), \tag{III.165}$$

which allows for constraints on the control input. It should be noted that the problems (ITP^e) and $(ITP^{c,e})$ are both minimization problems despite the fact that they

correspond to maximizing the expected number of targets detected. This is because the probability that at least one of the searchers detects the l^{th} target during $[0,1]$ is given by $1 - f^l(\eta)$, and hence the expected number of targets detected during $[0,1]$ is given by $\sum_{l=1}^{L} \left[1 - f^l(\eta)\right]$.

To formulate the problem of minimizing the probability that all of the serachers fail to detect any of the targets during $[0,1]$, we define the objective function $\psi^p : \mathbf{H} \to \mathbb{R}$ for any $\eta \in \mathbf{H}$ by

$$\psi^p(\eta) \triangleq \prod_{l=1}^{L} f^l(\eta). \tag{III.166}$$

Then, we consider the problems

$$(ITP^p) \quad \min_{\eta \in \mathbf{H}^0} \quad \psi^p(\eta), \tag{III.167}$$

and

$$(ITP^{c,p}) \quad \min_{\eta \in \mathbf{H}_c} \quad \psi^p(\eta). \tag{III.168}$$

2. Optimality Conditions

In this section, we state our assumptions and give optimality conditions for (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$. We begin by deriving parameterized differential equations of the augmented dynamics in terms of the augmented state, $\tilde{x}^l(t;\alpha^l)$, defined in (III.161). For all $l = 1, 2, ..., L$ and $t \in [0,1]$ we define

$$\tilde{h}^l(x(t), u(t); \alpha^l) \triangleq \begin{pmatrix} h^1(x^1(t), u^1(t)) \\ \vdots \\ h^K(x^K(t), u^K(t)) \\ \sum_{k=1}^{K} r^{k,l}(x^k(t), y^l(t;\alpha^l)) \end{pmatrix} \in \mathbb{R}^{nK+1}. \tag{III.169}$$

For a given $\alpha^l \in A$, $l = 1, 2, ..., L$, we write the following parameterized differential equation to describe the augmented dynamics

$$\dot{\tilde{x}}^l(t) = \tilde{h}^l(x(t), u(t); \alpha^l),\ t \in [0,1], \tilde{x}^l(0) = \tilde{\xi}. \tag{III.170}$$

We let $\tilde{x}^{\eta,l}(\cdot)$ denote the solution of (III.170) when the input is $\eta = (\xi, u)$, and $\tilde{\xi} = (\xi^T, 0)^T$. We next state a series of assumptions similar to those given in Section III.B.2, beginning with those related to $\phi^l(\cdot)$ and $y^l(\cdot;\cdot)$.

Assumption III.31. *We assume that $\phi^l(\cdot)$, $l = 1, 2, ..., L$ is four times continuously differentiable.* □

Assumption III.32. *We assume that Assumption III.2 holds, with α replaced by α^l, for $l = 1, 2, ..., L$.* □

The next assumption is related to $r^{k,l}(\cdot,\cdot)$ and $\tilde{h}^l(\cdot,\cdot;\cdot)$, where we will adopt the notation

$$\tilde{h}_x^l(x(t), u(t); \alpha^l) \triangleq \begin{pmatrix} h_x^1(x(t), u(t))^T \\ \vdots \\ h_x^K(x(t), u(t))^T \\ \sum_{k=1}^K \nabla_x r^{k,l}(x^k(t), y^l(t; \alpha^l))^T \end{pmatrix}, \tag{III.171}$$

where $\tilde{h}_x^l(x(t), u(t); \alpha^l)$ is a $(nK + 1) \times n$ matrix and

$$\tilde{h}_u^l(x(t), u(t); \alpha^l) \triangleq \begin{pmatrix} h_u^1(x(t), u(t))^T \\ \vdots \\ h_u^K(x(t), u(t))^T \\ 0 \end{pmatrix}, \tag{III.172}$$

where $\tilde{h}_u^l(x(t), u(t); \alpha^l)$ is a $(nK + 1) \times m$ matrix.

Assumption III.33. *We assume that Assumption III.3 holds with $\tilde{h}(\cdot,\cdot;\alpha)$, $\tilde{h}_x(\cdot,\cdot;\alpha)$, and $\tilde{h}_u(\cdot,\cdot;\alpha)$ replaced by $\tilde{h}^l(\cdot,\cdot;\alpha^l)$, $\tilde{h}_x^l(\cdot,\cdot;\alpha^l)$, and $\tilde{h}_u^l(\cdot,\cdot;\alpha^l)$, respectively.* □

We next show that $\psi^e(\cdot)$ and $\psi^p(\cdot)$ are Gateaux differentiable on $\mathbf{H}^0$.

Proposition III.34. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied. Then, for any $\eta \in \mathbf{H}^0$ and $\delta\eta \in H_{\infty,2}$*

(a) $\psi^e(\cdot)$ has a Gateaux differential $D\psi^e(\eta; \delta\eta)$ at η given by

$$D\psi^e(\eta; \delta\eta) = \langle \nabla\psi^e(\eta), \delta\eta \rangle_{H_2}, \tag{III.173}$$

where the gradient $\nabla\psi^e(\eta)$ is given by

$$\nabla\psi^e(\eta)(t) = \sum_{l=1}^{L}\left(\int_{\alpha^l\in A}\nabla_\eta\tilde{f}^l(\eta;\alpha^l)(t)\phi^l(\alpha^l)d\alpha^l\right), \forall t\in[0,1], \qquad \text{(III.174)}$$

and

(b) $\psi^p(\cdot)$ has a Gateaux differential $D\psi^p(\eta;\delta\eta)$ at η given by

$$D\psi^p(\eta;\delta\eta) = \langle\nabla\psi^p(\eta),\delta\eta\rangle_{H_2}, \qquad \text{(III.175)}$$

where the gradient $\nabla\psi^p(\eta)$ is given by

$$\begin{aligned}&\nabla\psi^p(\eta)(t)\\ =\ & \sum_{l=1}^{L}\left(\left[\int_{\alpha^l\in A}\nabla_\eta\tilde{f}^l(\eta;\alpha^l)(t)\phi^l(\alpha^l)d\alpha^l\right]\left[\prod_{j=1|j\neq l}^{L}\int_{\alpha^j\in A}\tilde{f}^j(\eta;\alpha^j)\phi^j(\alpha^j)d\alpha^j\right]\right),\\ &\forall t\in[0,1]. \qquad \text{(III.176)}\end{aligned}$$

Proof. Part (a) follows by the same arguments as those used in the proof of Proposition III.5, with $\tilde{f}(\cdot;\alpha)$ replaced by $\tilde{f}^l(\cdot;\alpha^l)$, and the recognition that the Gateaux derivative of a sum is the sum of the Gateaux derivatives. Part (b) follows in a similar manner, with an application of a product rule for Gateaux derivatives. □

As in Section III.B.2, we again observe that Proposition III.34 also holds under weaker assumptions on $\phi^l(\cdot)$, but because we need Assumption III.31 later we adopt it here as well. This issue regarding Assumption III.31 also applies elsewhere in this chapter. Our next task is to show that $\nabla\psi^e(\cdot)$ and $\nabla\psi^p(\cdot)$ are Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$.

Lemma III.35. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied, then*

(a) the gradient $\nabla\psi^e(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$, and

(b) the gradient $\nabla\psi^p(\cdot)$ is Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$.

Proof. By Lemma III.9, sums and products of Lipschitz $\mathbf{H}$-continuous functions are Lipschitz $\mathbf{H}$-continuous on bounded subsets of $\mathbf{H}^0$. Based on Lemma III.6 we know

that the components of (III.174) and (III.176) are all Lipschitz $\mathbf{H}$-continuous, and therefore the entire summations in (III.174) and (III.176) are Lipschitz $\mathbf{H}$-continuous. $\square$

We define nonpositive optimality functions $\theta^e : \mathbf{H}^0 \to \mathbb{R}$, $\theta^{c,e} : \mathbf{H}_c \to \mathbb{R}$, $\theta^p : \mathbf{H}^0 \to \mathbb{R}$, and $\theta^{c,p} : \mathbf{H}_c \to \mathbb{R}$ by

$$\theta^e(\eta) \triangleq -\frac{1}{2}\|\nabla\psi^e(\eta)\|_{H_2}^2, \tag{III.177}$$

$$\theta^{c,e}(\eta) \triangleq \min_{\eta' \in \mathbf{H}_c} \langle \nabla\psi^e(\eta), \eta' - \eta \rangle_{H_2} + \frac{1}{2} \|\eta' - \eta\|_{H_2}^2 , \tag{III.178}$$

$$\theta^p(\eta) \triangleq -\frac{1}{2}\|\nabla\psi^p(\eta)\|_{H_2}^2, \tag{III.179}$$

and

$$\theta^{c,p}(\eta) \triangleq \min_{\eta' \in \mathbf{H}_c} \langle \nabla\psi^p(\eta), \eta' - \eta \rangle_{H_2} + \frac{1}{2} \|\eta' - \eta\|_{H_2}^2 , \tag{III.180}$$

which define optimality conditions for (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$, respectively.

Proposition III.36. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied.*

(a) $\theta^e(\cdot)$, $\theta^{c,e}(\cdot)$, $\theta^p(\cdot)$, and $\theta^{c,p}(\cdot)$ are $\mathbf{H}^0$-continuous functions.

(b) If $\hat{\eta} \in \mathbf{H}^0$ is a local minimizer of (ITP^e), then $\theta^e(\hat{\eta}) = 0$.

(c) If $\hat{\eta} \in \mathbf{H}_c$ is a local minimizer of $(ITP^{c,e})$, then $\theta^{c,e}(\hat{\eta}) = 0$.

(d) If $\hat{\eta} \in \mathbf{H}^0$ is a local minimizer of (ITP^p), then $\theta^p(\hat{\eta}) = 0$.

(e) If $\hat{\eta} \in \mathbf{H}_c$ is a local minimizer of $(ITP^{c,p})$, then $\theta^{c,p}(\hat{\eta}) = 0$.

Proof. The proof follows the same arguments as those for the proof of Theorem 4.2.3 in Polak (1997), with Proposition III.34 taking the place of Corollary 5.6.9 from Polak (1997) and Lemma III.35 taking the place of Theorem 4.1.3 from Polak (1997). $\square$

3. Consistent Approximations

In this section we define the approximating problems (ITP_N^e), $(ITP_N^{c,e})$, (ITP_N^p), and $(ITP_N^{c,p})$, and present consistency conditions for them. As in the case of dependent targets, we divide our development into two subsections. Both subsections develop consistent approximations for the pairs $((ITP^e), \theta^e)$, $((ITP^{c,e}), \theta^{c,e})$, $((ITP^p), \theta^p)$, and $((ITP^{c,p}), \theta^{c,p})$, but the first subsection only deals with time discretization while the second subsection considers time and space discretization.

a. Time-Discretized Problems

We again consider the approximate solution of (II.22) by means of forward Euler's method, which was given in (III.60). Simultaneously, we approximately solve (III.158) also by forward Euler's method. For any $\eta = (\xi, u) \in H_N$, $\alpha^l \in A$, and $N \in \mathcal{N}$, we set $z_N^{\eta,l}(0;\alpha^l) = 0$, $l = 1, 2, ..., L$, and for any $j = 0, 1, ..., N-1$,

$$z_N^{\eta,l}\left((j+1)/N;\alpha^l\right) - z_N^{\eta,l}\left(j/N;\alpha^l\right) = \frac{1}{N}\sum_{k=1}^{K} r^{k,l}\left(x_N^{\eta,k}(j/N), y^l(j/N;\alpha^l)\right). \quad \text{(III.181)}$$

Using the discretized "information state" given by the recursion (III.181), we define the approximate objective functions $\psi^{e,N} : \mathbf{H}_N \to \mathbb{R}$ and $\psi^{p,N} : \mathbf{H}_N \to \mathbb{R}$ for any $\eta \in \mathbf{H}_N$ and $N \in \mathcal{N}$ by

$$\psi_N^e(\eta) \triangleq \sum_{l=1}^{L} \int_{\alpha^l \in A} \exp\left(-z_N^{\eta,l}(1;\alpha^l)\right) \phi^l(\alpha^l) d\alpha^l, \quad \text{(III.182)}$$

and

$$\psi_N^p(\eta) \triangleq \prod_{l=1}^{L} \int_{\alpha^l \in A} \exp\left(-z_N^{\eta,l}(1;\alpha^l)\right) \phi^l(\alpha^l) d\alpha^l. \quad \text{(III.183)}$$

We also define the individual components of (III.182) and (III.183) by

$$f_N^l(\eta) \triangleq \int_{\alpha^l \in A} \exp\left(-z_N^{\eta,l}(1;\alpha^l)\right) \phi^l(\alpha^l) d\alpha^l, \forall l = 1, 2, ..., L. \quad \text{(III.184)}$$

Again, for the sake of notational simplification, for any $\alpha^l \in A$, $l = 1, 2, ..., L$, we also define the functions $\tilde{f}_N^l(\cdot;\alpha^l) : \mathbf{H}_N^0 \to \mathbb{R}$ by

$$\tilde{f}_N^l(\eta;\alpha^l) \triangleq F\left(\tilde{\xi}, \tilde{x}_N^{\eta,l}(t;\alpha^l)\right), \quad \text{(III.185)}$$

where F is as defined in (III.15) and $\tilde{x}_N^{\eta,l}(j/N;\alpha^l)$ is an augmented state defined by

$$\tilde{x}_N^{\eta,l}(j/N;\alpha^l) \triangleq \begin{pmatrix} x_N^{\eta}(j/N) \\ z_N^{\eta,l}(j/N;\alpha^l) \end{pmatrix} \in \mathbb{R}^{nK+1},\ j = 0, 1, ..., N-1. \tag{III.186}$$

Hence, for any $N \in \mathcal{N}$, we define the following approximating problems

$$(ITP_N^e) \quad \min_{\eta \in \mathbf{H}_N^0} \quad \psi_N^e(\eta), \tag{III.187}$$

$$(ITP_N^{c,e}) \quad \min_{\eta \in \mathbf{H}_{c,N}} \quad \psi_N^e(\eta), \tag{III.188}$$

$$(ITP_N^p) \quad \min_{\eta \in \mathbf{H}_N^0} \quad \psi_N^p(\eta), \tag{III.189}$$

and

$$(ITP_N^{c,p}) \quad \min_{\eta \in \mathbf{H}_{c,N}} \quad \psi_N^p(\eta). \tag{III.190}$$

We next consider the differentiability of $\psi_N^e(\cdot)$ and $\psi_N^p(\cdot)$.

Proposition III.37. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied, and $N \in \mathcal{N}$. Then, for any $\eta \in \mathbf{H}_N^0$ and $\delta\eta \in H_{\infty,2}$*

(a) $\psi_N^e(\cdot)$ has a Gateaux differential $D\psi_N^e(\eta;\delta\eta) = \langle \nabla\psi_N^e(\eta), \delta\eta \rangle_{H_2}$, where

$$\nabla\psi_N^e(\eta)(t) = \sum_{l=1}^{L} \int_{\alpha^l \in A} \nabla_\eta \tilde{f}_N^l(\eta;\alpha^l)(t)\phi^l(\alpha^l)d\alpha^l, \forall t \in [0,1], \tag{III.191}$$

and

(b) $\psi_N^p(\cdot)$ has a Gateaux differential $D\psi_N^p(\eta;\delta\eta) = \langle \nabla\psi_N^p(\eta), \delta\eta \rangle_{H_2}$, where

$$\begin{aligned} &\nabla\psi_N^p(\eta)(t) \\ = \;& \sum_{l=1}^{L} \left(\left[\int_{\alpha^l \in A} \nabla_\eta \tilde{f}_N^l(\eta;\alpha^l)(t)\phi^l(\alpha^l)d\alpha^l \right] \left[\prod_{j=1|j\neq l}^{L} \int_{\alpha^j \in A} \tilde{f}_N^j(\eta;\alpha^j)\phi^j(\alpha^j)d\alpha^j \right] \right), \\ & \forall t \in [0,1]. \end{aligned} \tag{III.192}$$

Proof. Part (a) can be proven by the same arguments used in the proof of Proposition III.5, with $\tilde{f}(\cdot;\alpha)$ replaced by $\tilde{f}^l_N(\cdot;\alpha^l)$, and the recognition that the Gateaux derivative of a sum is the sum of the Gateaux derivatives. Part (b) follows in a similar manner, with an application of a product rule for Gateaux derivatives. $\square$

Next we show that $\nabla\psi^e_N(\cdot)$ and $\nabla\psi^p_N(\cdot)$ are Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}^0_N$.

Lemma III.38. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied, then*

(a) the gradient $\nabla\psi^e_N(\cdot)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}^0_N$, and

(b) the gradient $\nabla\psi^p_N(\cdot)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}^0_N$.

Proof. By Lemma III.9, sums and products of Lipschitz $\mathbf{H}_N$-continuous functions are Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}^0_N$. Based on Lemma III.10(b) we know that the components of (III.191) and (III.192) are all Lipschitz $\mathbf{H}_N$-continuous, and therefore the entire summations in (III.191) and (III.192) are Lipschitz $\mathbf{H}_N$-continuous. $\square$

For any $N \in \mathcal{N}$, we define nonpositive optimality functions $\theta^e_N : \mathbf{H}^0_N \to \mathbb{R}$, $\theta^{c,e}_N : \mathbf{H}_{c,N} \to \mathbb{R}$, $\theta^p_N : \mathbf{H}^0_N \to \mathbb{R}$, and $\theta^{c,p}_N : \mathbf{H}_{c,N} \to \mathbb{R}$ by

$$\theta^e_N(\eta) \triangleq -\frac{1}{2}\|\nabla\psi^e_N(\eta)\|^2_{H_2}, \tag{III.193}$$

$$\theta^{c,e}_N(\eta) \triangleq \min_{\eta' \in \mathbf{H}_{c,N}} \langle \nabla\psi^s_N(\eta), \eta' - \eta\rangle_{H_2} + \frac{1}{2}\|\eta' - \eta\|^2_{H_2}, \tag{III.194}$$

$$\theta^p_N(\eta) \triangleq -\frac{1}{2}\|\nabla\psi^p_N(\eta)\|^2_{H_2}, \tag{III.195}$$

and

$$\theta^{c,p}_N(\eta) \triangleq \min_{\eta' \in \mathbf{H}_{c,N}} \langle \nabla\psi^p_N(\eta), \eta' - \eta\rangle_{H_2} + \frac{1}{2}\|\eta' - \eta\|^2_{H_2}, \tag{III.196}$$

which characterize stationary points of (ITP^e_N), $(ITP^{c,e}_N)$, (ITP^p_N), and $(ITP^{c,p}_N)$, respectively.

Proposition III.39. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied.*

(a) $\theta^e_N(\cdot)$, $\theta^{c,e}_N(\cdot)$, $\theta^p_N(\cdot)$, and $\theta^{c,p}_N(\cdot)$ are $\mathbf{H}^0_N$-continuous functions.

(b) If $\hat{\eta} \in \mathbf{H}^0_N$ is a local minimizer of (ITP^e_N), then $\theta^e_N(\hat{\eta}) = 0$.

(c) If $\hat{\eta} \in \mathbf{H}_{c,N}$ is a local minimizer of $(ITP^{c,e}_N)$, then $\theta^{c,e}_N(\hat{\eta}) = 0$.

(d) If $\hat{\eta} \in \mathbf{H}^0_N$ is a local minimizer of (ITP^p_N), then $\theta^p_N(\hat{\eta}) = 0$.

(e) If $\hat{\eta} \in \mathbf{H}_{c,N}$ is a local minimizer of $(ITP^{c,p}_N)$, then $\theta^{c,p}_N(\hat{\eta}) = 0$.

Proof. The proof follows the same arguments as the proof of Proposition 1.1.6 in Polak (1997), with the norms and inner products replaced with their H_2 equivalents. □

As is the case in Section III.B.3a with epi-convergence of (GTP_N) to (GTP), it is not possible to establish epi-convergence of (ITP^e_N) to (ITP^s) or (ITP^p_N) to (ITP^p). We define the problems

$$(ITP^e_{cl}) \quad \min_{\eta \in \mathbf{H}^0_{cl}} \quad \psi^e(\eta), \tag{III.197}$$

and

$$(ITP^p_{cl}) \quad \min_{\eta \in \mathbf{H}^0_{cl}} \quad \psi^p(\eta), \tag{III.198}$$

because it is possible to establish epi-convergence of (ITP^e_N) to (ITP^e_{cl}) and (ITP^p_N) to (ITP^p_{cl}). We will use the problems (ITP^e_{cl}) and (ITP^p_{cl}), with the following assumption.

Assumption III.40. *We assume that all local and global minimizers of* (ITP^e_{cl}) *and* (ITP^p_{cl}) *are in* $\mathbf{H}^0$. □

Again, in a manner similar to that of Section 3.3 of Polak (1997) we next show that the pairs $((ITP^e_N), \theta^e_N)$ in the sequence $\{((ITP^e_N), \theta^e_N)\}_{N \in \mathcal{N}}$ and the pairs $((ITP^p_N), \theta^p_N)$ in the sequence $\{((ITP^p_N), \theta^p_N)\}_{N \in \mathcal{N}}$ are consistent approximations for the pairs $((ITP^e_{cl}), \theta^e)$ and $((ITP^p_{cl}), \theta^p)$, respectively. In order to establish epi-convergence, we will need the following intermediate result.

Lemma III.41. *Suppose that* $\beta^l : \mathbf{H} \to [0,1]$ *and* $\beta_N^l : \mathbf{H}_N \to [0,1]$, $l = 1, 2, ..., L$, $N \in \mathbb{N}$, *are such that for all* $\eta \in \mathbf{H}$, $l = 1, 2, ..., L$, *and* $N \in \mathbb{N}$,

$$\left|\beta_N^l(\eta) - \beta^l(\eta)\right| \leq \Delta(N) \tag{III.199}$$

where $\Delta : \mathbb{N} \to [0, \infty)$ *is a strictly decreasing function with* $\Delta(N) \to 0$, *as* $N \to \infty$. *Then for all* $N \in \mathbb{N}$ *and* $\eta \in \mathbf{H}$,

(a)

$$\left|\sum_{l=1}^{L} \beta_N^l(\eta) - \sum_{l=1}^{L} \beta^l(\eta)\right| \leq L\Delta(N) \tag{III.200}$$

and

(b) there is a constant $K_\pi < \infty$ such that

$$\left|\prod_{l=1}^{L} \beta_N^l(\eta) - \prod_{l=1}^{L} \beta^l(\eta)\right| \leq K_\pi \Delta(N). \tag{III.201}$$

Proof. To prove part (a), we expand the summations in (III.200), combine terms, and use the triangle inequality to write

$$\begin{aligned}\left|\sum_{l=1}^{L} \beta_N^l(\eta) - \sum_{l=1}^{L} \beta^l(\eta)\right| &= \left|\sum_{l=1}^{L} \left(\beta_N^l(\eta) - \beta^l(\eta)\right)\right| \\ &\leq \sum_{l=1}^{L} \left|\beta_N^l(\eta) - \beta^l(\eta)\right| \\ &\leq L\Delta(N) \end{aligned} \tag{III.202}$$

For the proof of part (b), we write out both the products in (III.201) and express $\beta_N^l(\eta)$ in terms of $\beta^l(\eta)$ and $\Delta(N)$, then because $\beta^l(\eta) \in [0,1]$ and $\beta_N^l(\eta) \in [0,1]$ for all l we have that

$$\left|\prod_{l=1}^{L} \beta_N^l(\eta) - \prod_{l=1}^{L} \beta^l(\eta)\right|$$

$$\leq \prod_{l=1}^{L} \left(\beta^l(\eta) + \Delta(N)\right) - \prod_{l=1}^{L} \beta^l(\eta) \tag{III.203}$$

$$\leq \sum_{l=1}^{L} \binom{L}{l} [\Delta(N)]^l, \tag{III.204}$$

where the final inequality follows from taking the product of the $\left(\beta^l(\eta) + \Delta(N)\right)$ in (III.203) and combining like terms. Then because $\Delta(N) \to 0$ as $N \to \infty$, there exists $\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$ there is a constant $K_\pi < \infty$ such that

$$\sum_{l=1}^{L} \binom{L}{l} [\Delta(N)]^l = L\Delta(N) + \sum_{l=2}^{L} \binom{L}{l} [\Delta(N)]^l \leq K_\pi \Delta(N). \tag{III.205}$$

□

We now show the epi-convergence of (ITP_N^e) to (ITP_{cl}^e) and (ITP_N^p) to (ITP_{cl}^p).

Theorem III.42. *Suppose that Assumptions III.31, III.32, III.33, and III.40 are satisfied. Then*

(a) (ITP_N^e) epi-converges to (ITP_{cl}^e), as $N \to \infty$, and

(b) (ITP_N^p) epi-converges to (ITP_{cl}^p), as $N \to \infty$.

Proof. The proof follows the same arguments as the proof of Theorem III.17. We then invoke Lemma III.41(a) and (b) to complete the proofs of parts (a) and (b), respectively.

□

In order to show consistency of approximation for the pairs $((ITP_N^e),\ \theta_N^e)$, $((ITP_N^{c,e}),\ \theta_N^{c,e})$, $((ITP_N^p),\ \theta_N^p)$, and $((ITP_N^{c,p}),\ \theta_N^{c,p})$ we will need the following two results.

Lemma III.43. *Suppose that there exist constants, $C^l < \infty$ and $C_N^l < \infty$, and that the functions $\nabla\beta^l : \mathbf{H}^0 \to \mathbf{H}$, $\nabla\beta_N^l : \mathbf{H}_N^0 \to \mathbf{H}_N$, $\beta^l : \mathbf{H} \to [0,1]$, and $\beta_N^l : \mathbf{H}_N \to [0,1]$, $l = 1, 2, ..., L$, $N \in \mathbb{N}$, satisfy*

$$\begin{aligned}
\|\nabla\beta^l(\eta)\|_{H_2} &\leq C^l,\ \forall l = 1,2,...,L, \eta \in \mathbf{H}^0 && \text{(III.206)}\\
\|\nabla\beta_N^l(\eta)\|_{H_2} &\leq C_N^l,\ \forall l = 1,2,...,L, \eta \in \mathbf{H}_N^0, N \in \mathbb{N} && \text{(III.207)}\\
\left|\beta_N^l(\eta) - \beta^l(\eta)\right| &\leq \Delta(N),\ \forall l = 1,2,...,L, \eta \in \mathbf{H}_N, N \in \mathbb{N} && \text{(III.208)}
\end{aligned}$$

and

$$\left\|\nabla\beta_N^l(\eta) - \nabla\beta^l(\eta)\right\|_{H_2} \leq \Delta(N),\ \forall l = 1,2,...,L, \eta \in \mathbf{H}_N^0, N \in \mathbb{N}, \tag{III.209}$$

where $\Delta : \mathbb{N} \to [0,\infty)$ is a strictly decreasing function with $\Delta(N) \to 0$, as $N \to \infty$. Then,

(a) for all $N \in \mathbb{N}$ and $\eta \in \mathbf{H}_N^0$

$$\left\| \sum_{l=1}^{L} \nabla \beta_N^l(\eta) - \sum_{l=1}^{L} \nabla \beta^l(\eta) \right\|_{H_2} \leq L\sqrt{2}\Delta(N), \tag{III.210}$$

and

(b) there is a constant $K_{\pi G} < \infty$ such that for all $N \in \mathbb{N}$ and $\eta \in \mathbf{H}_N^0$

$$\left\| \sum_{l=1}^{L} \nabla \beta_N^l(\eta) \prod_{j=1|j\neq l}^{L} \beta_N^j(\eta) - \sum_{l=1}^{L} \nabla \beta^l(\eta) \prod_{j=1|j\neq l}^{L} \beta^j(\eta) \right\|_{H_2} \leq K_{\pi G}\Delta(N). \tag{III.211}$$

Proof. We begin by proving part (a). We know that

$$\left\| \sum_{l=1}^{L} \nabla \beta_N^l(\eta) - \sum_{l=1}^{L} \nabla \beta^l(\eta) \right\|_{H_2} \leq \sum_{l=1}^{L} \left\| \nabla \beta_N^l(\eta) - \nabla \beta^l(\eta) \right\|_{H_2}. \tag{III.212}$$

Let $l = 1, 2, ..., L$, $N \in \mathbb{N}$, and $\eta \in \mathbf{H}_N^0$ be arbitrary, then based on the definition of the H_2 norm $\left\| \nabla \beta_N^l(\eta) - \nabla \beta^l(\eta) \right\|_{H_2} \leq \Delta(N)$ implies that

$$\begin{aligned} \left\| \nabla \beta_N^l(\eta) - \nabla \beta^l(\eta) \right\|_{H_2}^2 &= \left\| \nabla_\xi \beta_N^l(\eta) - \nabla_\xi \beta^l(\eta) \right\|^2 \\ &+ \int_0^1 \left\| \nabla_u \beta_N^l(\eta)(t) - \nabla_u \beta^l(\eta)(t) \right\|^2 dt \\ &\leq [\Delta(N)]^2 + [\Delta(N)]^2, \end{aligned} \tag{III.213}$$

where $\nabla_\xi \beta_N^l(\eta) \in \mathbb{R}^n$, $\nabla_\xi \beta^l(\eta) \in \mathbb{R}^n$, $\nabla_u \beta_N^l(\eta) \in \mathbf{U}$, and $\nabla_u \beta^l(\eta) \in \mathbf{U}$. Hence

$$\left\| \nabla \beta_N^l(\eta) - \nabla \beta^l(\eta) \right\|_{H_2} \leq \sqrt{2}\Delta(N), \forall l = 1, 2, ..., L, \tag{III.214}$$

which completes the proof of part (a).

For the proof of part (b), let $g_N^l(\eta) \triangleq \prod_{j=1|j\neq l}^{L} \beta_N^j(\eta)$ and $g^l(\eta) \triangleq \prod_{j=1|j\neq l}^{L} \beta^j(\eta)$. Then,

$$\left\| \sum_{l=1}^{L} \nabla \beta_N^l(\eta) \prod_{j=1|j\neq l}^{L} \beta_N^j(\eta) - \sum_{l=1}^{L} \nabla \beta^l(\eta) \prod_{j=1|j\neq l}^{L} \beta^j(\eta) \right\|_{H_2}$$

$$
\begin{aligned}
&= \left\| \sum_{l=1}^{L} g_N^l(\eta) \nabla \beta_N^l(\eta) - \sum_{l=1}^{L} g^l(\eta) \nabla \beta^l(\eta) \right\|_{H_2} \\
&\leq \sum_{l=1}^{L} \left\| g_N^l(\eta) \nabla \beta_N^l(\eta) - g^l(\eta) \nabla \beta^l(\eta) \right\|_{H_2}. \qquad \text{(III.215)}
\end{aligned}
$$

Based on the definition of the H_2 norm we have

$$
\left\| g_N^l(\eta) \nabla \beta_N^l(\eta) - g^l(\eta) \nabla \beta^l(\eta) \right\|_{H_2}^2 =
$$

$$
\left\| g_N^l(\eta) \nabla_\xi \beta_N^l(\eta) - g^l(\eta) \nabla_\xi \beta^l(\eta) \right\|^2 + \int_0^1 \left\| g_N^l(\eta) \nabla_u \beta_N^l(\eta)(t) - g^l(\eta) \nabla_u \beta^l(\eta)(t) \right\|^2 dt. \qquad \text{(III.216)}
$$

Based on Lemma III.41(b), there exist constants $K_g^l < \infty$, $l = 1, 2, ..., L$, such that $\forall N \in \mathbb{N}$, and $\forall \eta \in \mathbf{H}_N^0$

$$
\left| g_N^l(\eta) - g^l(\eta) \right| \leq K_g^l \Delta(N). \qquad \text{(III.217)}
$$

We then consider the first term on the right-hand side of (III.216), which can be rewritten

$$
\begin{aligned}
& \left\| g_N^l(\eta) \nabla_\xi \beta_N^l(\eta) - g^l(\eta) \nabla_\xi \beta^l(\eta) \right\|^2 \\
= \;& \left\| g^l(\eta) \nabla_\xi \beta_N^l(\eta) - g^l(\eta) \nabla_\xi \beta^l(\eta) + \left(g_N^l(\eta) - g^l(\eta) \right) \nabla_\xi \beta_N^l(\eta) \right\|^2 \\
\leq \;& g^l(\eta)^2 \left\| \nabla_\xi \beta_N^l(\eta) - \nabla_\xi \beta^l(\eta) \right\|^2 + \left\| \left(g_N^l(\eta) - g^l(\eta) \right) \nabla_\xi \beta_N^l(\eta) \right\|^2 \\
\leq \;& [\Delta(N)]^2 + \left(g_N^l(\eta) - g^l(\eta) \right)^2 \left\| \nabla_\xi \beta_N^l(\eta) \right\|^2 \\
\leq \;& [\Delta(N)]^2 + \left(K_g^l \right)^2 [\Delta(N)]^2 \left\| \nabla_\xi \beta_N^l(\eta) \right\|^2 \\
\leq \;& [\Delta(N)]^2 + \left(C_N^l \right)^2 \left(K_g^l \right)^2 [\Delta(N)]^2 \\
\leq \;& \left(1 + \left(C_N^l \right)^2 \left(K_g^l \right)^2 \right) [\Delta(N)]^2. \qquad \text{(III.218)}
\end{aligned}
$$

Next, we consider the second term in the sum on the right-hand side of (III.216), which can be rewritten

$$\begin{aligned}
& \int_0^1 \left\| g_N^l(\eta)\nabla_u\beta_N^l(\eta)(t) - g^l(\eta)\nabla_u\beta^l(\eta)(t)\right\|^2 dt \\
= \; & \int_0^1 \left\| g^l(\eta)\nabla_u\beta_N^l(\eta)(t) - g^l(\eta)\nabla_u\beta^l(\eta)(t) + \left(g_N^l(\eta) - g^l(\eta)\right)\nabla_u\beta_N^l(\eta)(t)\right\|^2 dt \\
\leq \; & g^l(\eta)^2 \int_0^1 \left\|\nabla_u\beta_N^l(\eta)(t) - \nabla_u\beta^l(\eta)(t)\right\|^2 dt + \int_0^1 \left\|\left(g_N^l(\eta) - g^l(\eta)\right)\nabla_u\beta_N^l(\eta)(t)\right\|^2 dt \\
\leq \; & [\Delta(N)]^2 + \int_0^1 \left(g_N^l(\eta) - g^l(\eta)\right)^2 \left\|\nabla_u\beta_N^l(\eta)(t)\right\|^2 dt \\
\leq \; & [\Delta(N)]^2 + \left(K_g^l\right)^2 [\Delta(N)]^2 \int_0^1 \left\|\nabla_u\beta_N^l(\eta)(t)\right\|^2 dt \\
\leq \; & [\Delta(N)]^2 + \left(\left(C_N^l\right)^2\left(K_g^l\right)^2\right)[\Delta(N)]^2 \leq \left(1 + \left(C_N^l\right)^2\left(K_g^l\right)^2\right)[\Delta(N)]^2 .
\end{aligned} \tag{III.219}$$

Hence

$$\left\| g_N^l(\eta)\nabla\beta_N^l(\eta) - g^l(\eta)\nabla\beta^l(\eta)\right\|_{H_2} \leq \sqrt{2\left(1 + \left(C_N^l\right)^2\left(K_g^l\right)^2\right)}\Delta(N), \forall l = 1, 2, ..., L. \tag{III.220}$$

Then $K_{\pi G} = \sum_{l=1}^{L}\sqrt{2\left(1 + \left(C_N^l\right)^2\left(K_g^l\right)^2\right)}$, and the proof is complete. □

Proposition III.44. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied, then for every bounded subset $S \subset \mathbf{H}^0$,*

(a) there exists a constant $K_{\Sigma F} < \infty$ such that, for any $N \in \mathcal{N}$ and $\eta \in S \cap H_N$

$$\|\nabla\psi_N^e(\eta) - \nabla\psi^e(\eta)\|_{H_2} \leq \frac{K_{\Sigma F}}{N} \tag{III.221}$$

and

(b) there exists a constant $K_{\pi F} < \infty$ such that, for any $N \in \mathcal{N}$ and $\eta \in S \cap H_N$

$$\|\nabla\psi_N^p(\eta) - \nabla\psi^p(\eta)\|_{H_2} \leq \frac{K_{\pi F}}{N}. \tag{III.222}$$

Proof. From Proposition III.16, we deduce that

$$\int_{\alpha^l \in A} \left|\tilde{f}_N^l(\eta;\alpha^l) - \tilde{f}^l(\eta;\alpha^l)\right| \phi^l(\alpha^l) d\alpha^l \leq \frac{K_S}{N}. \tag{III.223}$$

The proof then follows the same arguments as the proof of Proposition III.18 with Propositions III.5 and III.11 replaced by Propositions III.34 and III.37, respectively. We then invoke Lemma III.43(a) and (b) to complete the proofs of parts (a) and (b), respectively. □

We now show that the pairs $((ITP_N^e), \theta_N^e)$ in the sequence $\{((ITP_N^e), \theta_N^e)\}_{N\in\mathcal{N}}$ and the pairs $((ITP_N^p), \theta_N^p)$ in the sequence $\{((ITP_N^p), \theta_N^p)\}_{N\in\mathcal{N}}$ are consistent approximations for the pairs $((ITP_{cl}^e), \theta^e)$ and $((ITP_{cl}^p), \theta^p)$, respectively.

Theorem III.45. *Suppose that Assumptions III.31, III.32, III.33, and III.40 are satisfied, (ITP_{cl}^e), (ITP_{cl}^p), θ^e, θ^p, (ITP_N^e), (ITP_N^p), θ_N^e, and θ_N^p are defined as in (III.197), (III.198), (III.177), (III.179), (III.187), (III.189), (III.193), and (III.195), respectively. Then*

(a) the pairs $((ITP_N^e),\ \theta_N^e)$, in the sequence $\{((ITP_N^e), \theta_N^e)\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP_{cl}^e), \theta^e)$, and

(b) the pairs $((ITP_N^p),\ \theta_N^p)$, in the sequence $\{((ITP_N^p), \theta_N^p)\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^p cl), \theta^p)$.

Proof. The proof follows the same arguments as the proof of Theorem III.20 with Theorem III.17, Lemma III.6, and Proposition III.18 replaced by Theorem III.42, Lemma III.35, and Proposition III.44, respectively. □

We next show that the pairs $((ITP_N^{c,e}), \theta_N^{c,e})$ in the sequence $\{((ITP_N^{c,e}), \theta_N^{c,e})\}_{N\in\mathcal{N}}$ and the pairs $((ITP_N^{c,p}), \theta_N^{c,p})$ in the sequence $\{((ITP_N^{c,p}), \theta_N^{c,p})\}_{N\in\mathcal{N}}$ are consistent approximations for the pairs $((ITP^{c,e}), \theta^{c,e})$ and $((ITP^{c,p}), \theta^{c,p})$, respectively.

Theorem III.46. *Suppose that Assumptions III.31, III.32, III.33, and III.40 are satisfied, $(ITP^{c,e})$, $(ITP^{c,p})$, $\theta^{c,e}$, $\theta^{c,p}$, $(ITP_N^{c,e})$, $(ITP_N^{c,p})$, $\theta_N^{c,e}$, and $\theta_N^{c,p}$ are defined as in (III.165), (III.168), (III.178), (III.180), (III.188), (III.190), (III.194),and (III.196), respectively. Then*

(a) the pairs $((ITP_N^{c,e}),\ \theta_N^{c,e})$, in the sequence $\{((ITP_N^{c,e}), \theta_N^{c,e})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^{c,e}), \theta^{c,e})$, and

(b) the pairs $((ITP_N^{c,p}),\ \theta_N^{c,p})$, in the sequence $\{((ITP_N^{c,p}), \theta_N^{c,p})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^{c,p}), \theta^{c,p})$.

Proof. The proof follows the same arguments as the proof of Theorem III.21 with Theorems III.17 and III.20 replaced by Theorems III.42 and III.45, respectively. ☐

b. Time- and Space-Discretized Problems

We next consider the time- and space-discretized problem. We make use of the integration rule I_M given in (III.102) to define the approximate objective functions $\psi^e_{NM} : \mathbf{H}_N \to \mathbb{R}$ and $\psi^p_{NM} : \mathbf{H}_N \to \mathbb{R}$ for any $\eta \in \mathbf{H}_N$, $N \in \mathcal{N}$, and $M \in \mathbb{N} \times \mathbb{N}$ by

$$\psi^e_{NM}(\eta) \triangleq \sum_{l=1}^{L} I_M\left(\exp\left[-z_N^{\eta;l}(1;\cdot)\right]\phi^l(\cdot)\right), \tag{III.224}$$

and

$$\psi^p_{NM}(\eta) \triangleq \prod_{l=1}^{L} I_M\left(\exp\left[-z_N^{\eta;l}(1;\cdot)\right]\phi^l(\cdot)\right). \tag{III.225}$$

We next consider the differentiability of $\psi^e_{NM}(\cdot)$ and $\psi^p_{NM}(\cdot)$.

Proposition III.47. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied, $N \in \mathcal{N}$, and $M \in \mathbb{N} \times \mathbb{N}$, then for any $\eta \in \mathbf{H}_N^0$ and $\delta\eta \in H_{\infty,2}$,*

(a) $\psi^e_{NM}(\cdot)$ has a Gateaux differential $D\psi^e_{NM}(\eta;\delta\eta) = \langle \nabla\psi^e_{NM}(\eta), \delta\eta\rangle_{H_2}$, where

$$\nabla\psi^e_{NM}(\eta)(t) = \sum_{l=1}^{L}\left(I_M\left[\nabla_\eta \tilde{f}_N^l(\eta;\cdot)(t)\phi^l(\cdot)\right]\right), \forall t \in [0,1], \tag{III.226}$$

and

(b) $\psi^p_{NM}(\cdot)$ has a Gateaux differential $D\psi^p_{NM}(\eta;\delta\eta) = \langle \nabla\psi^p_{NM}(\eta), \delta\eta\rangle_{H_2}$, where

$$\nabla\psi^p_{NM}(\eta)(t) = \sum_{l=1}^{L}\left[I_M\left[\nabla_\eta \tilde{f}_N^l(\eta;\cdot)(t)\phi^l(\cdot)\right]\right]\left[\prod_{j=1|j\neq l}^{L} I_M\left[\tilde{f}_N^j(\eta;\cdot)\phi^j(\cdot)\right]\right],$$
$$\forall t \in [0,1]. \tag{III.227}$$

Proof. The proof of part (a) follows the same arguments used in the proof of Proposition III.22, with $\tilde{f}(\cdot;\alpha)$ replaced by $\tilde{f}_N^l(\cdot;\alpha^l)$, and with the recognition that the Gateaux derivative of a sum is the sum of the Gateaux derivatives. Part (b) follows in a similar manner, with an application of a product rule for Gateaux derivatives. ☐

Next, we show that $\nabla\psi^e_{NM}(\cdot)$ and $\nabla\psi^p_{NM}(\cdot)$ are Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}_N^0$.

Lemma III.48. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied, then*

(a) the gradient $\nabla\psi^e_{NM}(\eta)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}^0_N$, and

(b) the gradient $\nabla\psi^p_{NM}(\eta)$ is Lipschitz $\mathbf{H}_N$-continuous on bounded subsets of $\mathbf{H}^0_N$.

Proof. The proof follows the same arguments as the proof of Lemma III.38, with integration replaced by I_M. □

For the reasons discussed in Section III.B.3b, in order to conduct the analysis that follows we again make the assumptions delineated in Assumption III.24. Based on the choice of Composite Simpson's rule as the numerical integration scheme, we find it necessary to show that the partial derivatives of $\tilde{f}^l_N(\eta;\cdot)\phi^l(\cdot)$ and $\nabla_\eta\tilde{f}^l_N(\eta;\cdot)\phi^l(\cdot)$ up to and including the fourth-order are bounded for any choice of $\eta \in \mathbf{H}^0_N$, any $\alpha^l \in A$, and any $N \in \mathcal{N}$. The bounding of the partial derivatives of $\tilde{f}^l_N(\eta;\cdot)\phi^l(\cdot)$ and $\nabla_\eta\tilde{f}^l_N(\eta;\cdot)\phi^l(\cdot)$ is necessary in order to prove the convergence of $\psi^e_{NM}(\eta)$ to $\psi^e(\eta)$, $\nabla\psi^e_{NM}(\eta)$ to $\nabla\psi^e(\eta)$, $\psi^p_{NM}(\eta)$ to $\psi^p(\eta)$, and $\nabla\psi^p_{NM}(\eta)$ to $\nabla\psi^p(\eta)$. To facilitate these proofs we begin by defining some notation. For any $\eta \in \mathbf{H}^0_N$, $\alpha^l \in A$, $l = 1, 2, ..., L$, $N \in \mathcal{N}$, and $j = 0, 1, ..., N-1$, we define

$$\zeta^l_1(\alpha^l) \triangleq \exp\left[-\sum_{j=0}^{N-1}\frac{1}{N}\sum_{k=1}^{K} r^{k,l}\left(x^{\eta,k}_N\left(\frac{j}{N}\right), y^l\left(\frac{j}{N};\alpha^l\right)\right)\right], \tag{III.228}$$

$$\zeta^l_2(\alpha^l) \triangleq \phi^l(\alpha^l), \tag{III.229}$$

$$\zeta^l_3(\alpha^l) \triangleq p^{\eta,l}_N(0;\alpha^l), \tag{III.230}$$

and

$$\zeta^l_4(\alpha^l) \triangleq p^{\eta,l}_N\left(\frac{j+1}{N};\alpha^l\right). \tag{III.231}$$

We again note that $\zeta^l_1(\cdot)$, $\zeta^l_3(\cdot)$, and $\zeta^l_4(\cdot)$, $l = 1, 2, ..., L$, depend on η and N.

We next show that the partial derivatives of $\zeta_1^l(\cdot), ..., \zeta_4^l(\cdot)$ up to and including the fourth-order are continuous and bounded for any choice of $\eta \in \mathbf{H}_N^0$, $\alpha^l \in A$, $l = 1, 2, ..., L$, and $N \in \mathcal{N}$.

Lemma III.49. *Suppose that Assumptions III.31, III.32, and III.33 are satisfied and S is a bounded subset of $\mathbf{H}_N^0$. Then,*

(i) $\zeta_i^l(\cdot) \in \mathcal{C}^4(A) \ \ \forall i = 1, 2, 3, 4,$

and

(ii) there exists $C < \infty$, such that for all $\eta \in S$, $j = 0, 1, ..., N-1$, $\alpha^l \in A$, $l = 1, 2, ..., L$, and $N \in \mathcal{N}$

$$\left| \frac{\partial^\mu \zeta_\kappa^l(\alpha^l)}{\partial \alpha_i^{l,\mu}} \right| \leq C \ \ \forall i = 1, 2, \forall \mu = 1, 2, 3, 4, \forall \kappa = 1, 2, 3, 4. \tag{III.232}$$

Proof. The proof follows the same arguments as the proof of Lemma III.25 with Assumptions III.1, III.2, and III.3 replaced by Assumptions III.31, III.32, and III.33, respectively. □

In order to prove epi-convergence and consistency of approximation, we will need the following proposition.

Proposition III.50. *Suppose that Assumptions III.24, III.31, III.32, and III.33 are satisfied. Then for every bounded subset $S \subset \mathbf{H}^0$, there exist constants $K_{\Sigma S} < \infty$, $K_{\pi S} < \infty$, and $K_{\pi I1}, K_{\pi I2} < \infty$ such that, for any $N \in \mathcal{N}$, for any $M \in \mathbb{N}_3 \times \mathbb{N}_3$*[4]*, and $\eta \in S \cap H_N$,*

(a) (i) $|\psi^e(\eta) - \psi^e_{NM}(\eta)| \leq \frac{K_{\Sigma S}}{N} + \frac{K_{\pi I1}}{(M_1-1)^4} + \frac{K_{\pi I2}}{(M_2-1)^4}$,

(ii) $\|\nabla\psi^e(\eta) - \nabla\psi^e_{NM}(\eta)\|_{H_2} \leq \frac{K_{\Sigma F}}{N} + \frac{K_{\pi I1}}{(M_1-1)^4} + \frac{K_{\pi I2}}{(M_2-1)^4}$,

and

(b) (i) $|\psi^p(\eta) - \psi^p_{NM}(\eta)| \leq \frac{K_{\pi S}}{N} + \frac{K_{\pi I1}}{(M_1-1)^4} + \frac{K_{\pi I2}}{(M_2-1)^4}$,

(ii) $\|\nabla\psi^p(\eta) - \nabla\psi^p_{NM}(\eta)\|_{H_2} \leq \frac{K_{\pi F}}{N} + \frac{K_{\pi I1}}{(M_1-1)^4} + \frac{K_{\pi I2}}{(M_2-1)^4}$,

and $K_{\Sigma F}$ and $K_{\pi F}$ are defined as in Proposition III.44.

[4]Recall that $\mathbb{N}_3 \triangleq \{m \in 2\mathbb{N}+1 | m \geq 3\}$, as defined previously in Proposition III.26.

Proof. The proof follows the same arguments as the proof of Proposition III.26 with Proposition III.18, Lemma III.25, and $\zeta_1(\cdot), ..., \zeta_4(\cdot)$ replaced by Proposition III.44, Lemma III.49, and $\zeta_1^l(\cdot), ..., \zeta_4^l(\cdot)$, respectively. $\square$

For any $N \in \mathcal{N}$, and $M \in \mathbb{N}\times\mathbb{N}$, we define the following approximating problems

$$(ITP^e_{NM}) \quad \min_{\eta\in\mathbf{H}^0_N} \quad \psi^e_{NM}(\eta), \tag{III.233}$$

$$(ITP^{c,e}_{NM}) \quad \min_{\eta\in\mathbf{H}_{c,N}} \quad \psi^e_{NM}(\eta), \tag{III.234}$$

$$(ITP^p_{NM}) \quad \min_{\eta\in\mathbf{H}^0_N} \quad \psi^p_{NM}(\eta), \tag{III.235}$$

and

$$(ITP^{c,p}_{NM}) \quad \min_{\eta\in\mathbf{H}_{c,N}} \quad \psi^p_{NM}(\eta). \tag{III.236}$$

For any $N \in \mathcal{N}$, and $M \in \mathbb{N} \times \mathbb{N}$, we define nonpositive optimality functions $\theta^e_{NM} : \mathbf{H}^0_N \to \mathbb{R}$, $\theta^{c,e}_{NM} : \mathbf{H}_{c,N} \to \mathbb{R}$, $\theta^p_{NM} : \mathbf{H}^0_N \to \mathbb{R}$, and $\theta^{c,p}_{NM} : \mathbf{H}_{c,N} \to \mathbb{R}$ by

$$\theta^e_{NM}(\eta) \triangleq -\frac{1}{2}\|\nabla\psi^e_{NM}(\eta)\|^2_{H_2}, \tag{III.237}$$

$$\theta^{c,e}_{NM}(\eta) \triangleq \min_{\eta'\in\mathbf{H}_{c,N}} \langle\nabla\psi^e_{NM}(\eta), \eta'-\eta\rangle_{H_2} + \frac{1}{2}\left\|\eta'-\eta\right\|^2_{H_2}, \tag{III.238}$$

$$\theta^p_{NM}(\eta) \triangleq -\frac{1}{2}\|\nabla\psi^p_{NM}(\eta)\|^2_{H_2}, \tag{III.239}$$

and

$$\theta^{c,p}_{NM}(\eta) \triangleq \min_{\eta'\in\mathbf{H}_{c,N}} \langle\nabla\psi^p_{NM}(\eta), \eta'-\eta\rangle_{H_2} + \frac{1}{2}\left\|\eta'-\eta\right\|^2_{H_2}, \tag{III.240}$$

which characterize stationary points of (ITP^e_{NM}), $(ITP^{c,e}_{NM})$, (ITP^p_{NM}), and $(ITP^{c,p}_{NM})$, respectively.

Proposition III.51. *Suppose that Assumptions III.24, III.32, and III.33 are satisfied.*

(a) $\theta^{e}_{NM}(\cdot)$, $\theta^{c,e}_{NM}(\cdot)$, $\theta^{p}_{NM}(\cdot)$, and $\theta^{c,p}_{NM}(\cdot)$ are $\mathbf{H}^0_N$-continuous functions.

(b) If $\hat{\eta} \in \mathbf{H}^0_N$ is a local minimizer of (ITP^e_{NM}), then $\theta^e_{NM}(\hat{\eta}) = 0$.

(c) If $\hat{\eta} \in \mathbf{H}_{c,N}$ is a local minimizer of $(ITP^{c,e}_{NM})$, then $\theta^{c,e}_{NM}(\hat{\eta}) = 0$.

(d) If $\hat{\eta} \in \mathbf{H}^0_N$ is a local minimizer of (ITP^p_{NM}), then $\theta^p_{NM}(\hat{\eta}) = 0$.

(e) If $\hat{\eta} \in \mathbf{H}_{c,N}$ is a local minimizer of $(ITP^{c,p}_{NM})$, then $\theta^{c,p}_{NM}(\hat{\eta}) = 0$.

Proof. The proof follows the same arguments as the proof of Proposition 1.1.6 in Polak (1997), with the norms and inner products replaced by their H_2 equivalents. □

It is not possible to establish epi-convergence of $(ITP^e_{NM(N)})$ to (ITP^e) or $(ITP^p_{NM(N)})$ to (ITP^p), but it is possible to establish epi-convergence of $(ITP^e_{NM(N)})$ to (ITP^e_{cl}) and $(ITP^p_{NM(N)})$ to (ITP^p_{cl}). We now show that the family $\{((ITP^e_{NM(N)}), \theta^e_{NM(N)})\}_{N\in\mathcal{N}}$ is a sequence of consistent approximations for $((ITP^e_{cl}), \theta^e)$, and the family $\{((ITP^p_{NM(N)}), \theta^p_{NM(N)})\}_{N\in\mathcal{N}}$ is a sequence of consistent approximations for $((ITP^p_{cl}), \theta^p)$ by showing that the conditions of Definition III.2 are satisfied.

Theorem III.52. *Suppose that Assumptions III.28, III.32, III.33, and III.40 are satisfied,* (ITP^e_{cl}), $(ITP^p cl)$, θ^e, θ^p, $(ITP^e_{NM(N)})$, $(ITP^p_{NM(N)})$, $\theta^e_{NM(N)}$, *and* $\theta^p_{NM(N)}$ *are defined as in (III.197), (III.198), (III.177), (III.179), (III.233), (III.235), (III.237), and (III.239), respectively. Then*

(a) the pairs $((ITP^e_{NM(N)}),\ \theta^e_{NM(N)})$, in the sequence $\{((ITP^e_{NM(N)}), \theta^e_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^e_{cl}), \theta^e)$, and

(b) the pairs $((ITP^p_{NM(N)}),\ \theta^p_{NM(N)})$, in the sequence $\{((ITP^p_{NM(N)}), \theta^p_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^p_{cl}), \theta^p)$.

Proof. The proof follows the same arguments as the proof of Theorem III.29 with Proposition III.26 and Theorems III.17 and III.20 replaced by Proposition III.50 and Theorems III.42 and III.45, respectively. □

We now show that the pairs $((ITP^{c,e}_{NM(N)}), \theta^{c,e}_{NM(N)})$ in the sequence $\{((ITP^{c,e}_{NM(N)}), \theta^{c,e}_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^{c,e}), \theta^{c,e})$.

We also show that the pairs $((ITP^{c,p}_{NM(N)}), \theta^{c,p}_{NM(N)})$ in the sequence $\{((ITP^{c,p}_{NM(N)}), \theta^{c,p}_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^{c,p}), \theta^{c,p})$.

Theorem III.53. *Suppose that Assumptions III.28, III.32, III.33, and III.40 are satisfied,* $(ITP^{c,e})$, $(ITP^{c,p})$, $\theta^{c,e}$, $\theta^{c,p}$, $(ITP^{c,e}_{NM(N)})$, $(ITP^{c,p}_{NM(N)})$, $\theta^{c,e}_{NM(N)}$, *and* $\theta^{c,p}_{NM(N)}$ *are defined as in (III.165), (III.168), (III.178), (III.180), (III.234), (III.236), (III.238), and (III.240), respectively. Then*

(a) the pairs $((ITP^{c,e}_{NM(N)}),\ \theta^{c,e}_{NM(N)})$, in the sequence $\{((ITP^{c,e}_{NM(N)}), \theta^{c,e}_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^{c,e}), \theta^{c,e})$, and

(b) the pairs $((ITP^{c,p}_{NM(N)}),\ \theta^{c,p}_{NM(N)})$, in the sequence $\{((ITP^{c,p}_{NM(N)}), \theta^{c,p}_{NM(N)})\}_{N\in\mathcal{N}}$ are consistent approximations for the pair $((ITP^{c,p}), \theta^{c,p})$.

Proof. The proof follows the same arguments as the proof of Theorem III.30, with Theorem III.29 replaced by Theorem III.52. $\square$

IV. RATE OF CONVERGENCE ANALYSIS

A. INTRODUCTION

In this chapter, we develop rate of convergence results by expressing the rate of convergence in terms of the computational work, rather than the typical number of iterations or level of discretization. We relate the compuational work to the number of iterations as well as to the levels of discretization by making computational work assumptions. This relation allows us to determine a guaranteed rate of convergence for the error between the objective function evaluated at iterates generated from the discretized problems and the optimal value of the original problem as a function of the computational work. Because these results are applicable to a range of approximation problems, we prefer to develop our results using abstract problems as we did in Section III.B.3. We again define the infinite-dimensional problem

$$(P) \quad \min_{x \in X} \quad f(x), \tag{IV.1}$$

where $f(\cdot)$ and X are defined as in Section III.B.3a. For all $N, M \in \mathbb{N}$, let $f_{NM} : \mathcal{B}_N \to \bar{\mathbb{R}}$ be a lower semicontinuous function that approximates $f(\cdot)$ on $\mathcal{B}_N$, and let $X_N \subset \mathcal{B}_N$ be an approximation to X, where $\mathcal{B}_N$ and $\bar{\mathbb{R}}$ are defined as in Section III.B.3a. We then define the family of finite dimensional approximating problems

$$(P_{NM}) \quad \min_{x \in X_N} \quad f_{NM}(x), \; N, M \in \mathcal{N}. \tag{IV.2}$$

We note that x is now a decision variable, and it no longer represents the physical state of the searcher. We also note that the problems (GTP), (GTP^c), (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$ discussed in Chapter III are examples of (P). The problems $(GTP_{NM}),(GTP^c_{NM})$, (ITP^e_{NM}), $(ITP^{c,e}_{NM})$, (ITP^p_{NM}), and $(ITP^{c,p}_{NM})$ discussed in Chapter III are examples of (P_{NM}) under the assumption that $M_1 = M_2 = M$. Although the examples we provide from Chapter III are all generalized optimal control problems, it is worth noting that the results of this chapter are applicable to other types of problems as well. The results may be applicable to approximation problems

for (P) that involve a single discretization parameter for the space of decision variables and two discretization parameters for the objective function.

To solve the infinite-dimensional problem (P) some form of discretization is necessary. If, for example, the infinite-dimensional problem takes the form of (GTP) and N and M represent time and space discretization levels, respectively, then the discretization in time of the searcher's dynamics using Euler's method and approximation of spatial integration via Simpson's rule described in Section III.B.3b combined with a standard nonlinear programming algorithm could be used to find a solution to (P). We refer to this type of solution methodology as a discretization algorithm.

In order to solve (P) using a discretization algorithm, there is a fundamental trade-off between the level of discretization and the computational work required to approximately solve the resulting finite-dimensional problem (P_{NM}). A fine level of N and M discretization ensures that (P_{NM}) approximates (P), in some sense, with a high degree of accuracy. The issue with this approach is that the function and gradient evaluations in (P_{NM}) tend to be expensive, and therefore the computational work required is high. A lower level of N and M discretization results in a faster solution time, due to the relatively less expensive function and gradient evaluations, but at the expense of a less accurate approximation of (P). It is usually difficult in practice to effectively manage the trade-off between these diametrically opposed goals of solution accuracy and computational cost.

In this chapter, we investigate the rate of convergence of a class of discretization algorithms as a computing budget tends to infinity. We show that the policy for selecting N and M discretization levels relative to the size of the available computational budget influences the rate of convergence of discretization algorithms. We identify optimal discretization policies, in a specifically defined sense, for discretization algorithms used to solve the resulting finite-dimensional problems based on finitely, superlinearly, linearly, and sublinearly convergent optimization algorithms.

Other researchers have examined the rate of convergence for optimization prob-

lems where discretization is necessary to obtain solutions. Dunn and Sachs (1983) considers the effect of approximate function and gradient evaluations on the convergence rates of the conditional gradient and projected gradient optimization algorithms. Kelley and Sachs (1986) examines the effect of approximating the Hessian on the convergence rate of the BFGS-method when it is used to solve discretized optimal control problems. Dupuis and James (1998) presents a method for obtaining rate of convergence estimates for finite difference approximation schemes for stochastic and deterministic optimal control problems. The method given in Dupuis and James (1998) requires some assumption about the smoothness of the objective function being approximated, which they exploit to obtain sharp rates and in some cases an expansion of the discretization error in terms of the discretization parameter. Kang (2008) gives rate of convergence results for the approximate optimal cost computed when using pseudospectral methods to solve optimal control problems. These studies all share two important similarities. The first being that the approximation schemes under consideration are all based on a single discretization parameter, and the second being that none of the studies consider the error introduced by the optimization algorithm in their determination of the rate of convergence.

More recent studies in the Monte Carlo simulation and simulation optimization literature (see Chen & Shi, 2008 for a summary) deal with how to optimally allocate a computational budget across different tasks within the simulation and to determine the resulting rate of convergence of an estimator as the computational budget tends to infinity. In the stochastic optimization literature, Pasupathy (2010) and Royset and Szechtman (2011) consider optimization algorithms with sublinear, linear, and superlinear rates of convergence for use with the sample average approximation approach, determine optimal policies for allocating a compuational budget between sampling and optimization, and quantify the associated rate of convergence of the sample average approximation approach as the computational budget tends to infinity. In the semi-infinite minimax literature, Royset and Pee (2011) examines the convergence

rate as the computing budget tends to infinity, and provide allocation polices that maximize the convergence rate of sublinear, linear, superlinear, and finite algorithms used to solve semi-infinite minimax problems. Royset and Pee (2011) also examines the solution of the semi-infinite minimax problem by exponential smoothing algorithms, provides an optimal discretization and smoothing policy, and determines the corresponding rate of convergence as the computing budget tends to infinity. We note that the section on smoothing algorithms in Royset and Pee (2011) uses two distinct discretization parameters. Our approach in this chapter follows the methodology of Royset and Szechtman (2011) and Royset and Pee (2011). Our analysis is based on different assumptions, however, and as a result we reach different conclusions.

The organization of this chapter is as follows. We begin by defining terminology and presenting our assumptions. Next we consider finite, superlinear, linear, and sublinear optimization algorithms for solving the finite-dimensional problem (P_{NM}). For each class of optimization algorithm we give an optimal discretization policy with corresponding rate of convergence, as the computational budget tends to infinity. Finally, we state our conclusions.

B. TERMINOLOGY AND ASSUMPTIONS

We begin with some assumptions about (P) and (P_{NM}).

Assumption IV.1. *We assume that the following hold:*

(i) The set of optimal solutions X^* of (P) is nonempty.

(ii) There exist constants $\bar{N}, \bar{M} \in \mathbb{N}$, and $K \in [0, \infty)$ such that

  (a) the set of optimal solutions X^*_{NM} of (P_{NM}) is nonempty for all $N \geq \bar{N}$, $M \geq \bar{M}$, $N, M \in \mathbb{N}$, and

  (b) there exist $p, q \in (0, \infty)$ such that

$$|f(x) - f_{NM}(x)| \leq \frac{K}{N^p} + \frac{K}{M^q}, \tag{IV.3}$$

  for all $x \in X_N$, $N \geq \bar{N}$, $M \geq \bar{M}$, and $N, M \in \mathbb{N}$. □

If, for example, (P_{NM}) is a generalized optimal control problem of the form defined in Chapter III, then part (b) of item (ii) holds when (P_{NM}) is solved using an algorithm that utilizes Euler's method to approximate the solutions of the differential equations and approximates spatial integration using Simpson's rule. In this case $p = 1$ and $q = 4$; see Proposition III.26 and Proposition III.50. We refer to

$$f(x) - f_{NM}(x) \tag{IV.4}$$

as the discretization error. Unless X_N and $f_{NM}(x)$, $x \in X_N$, have special structures, it is impossible to obtain a globally optimal solution of (P_{NM}) in finite computing time. Therefore, once a finite number of iterations of an optimization algorithm are applied to (P_{NM}), there is usually optimization error. Given an optimization algorithm $\mathcal{A}$ for (P_{NM}), let $x^n_{NM} \in X_N$ be the iterate obtained by $\mathcal{A}$ following n iterations on (P_{NM}). We use the notation f^*_{NM} to denote the optimal value of (P_{NM}) and define the optimization error as

$$f_{NM}(x^n_{NM}) - f^*_{NM}. \tag{IV.5}$$

The rate of decay of the optimization error as n gets larger depends on the rate of convergence of $\mathcal{A}$ when used to solve (P_{NM}). Throughout this dissertation we only consider deterministic algorithms that generate iterates in X exclusively, which we formalize in the next assumption.

Assumption IV.2. *We assume that if $\{x^n_{NM}\}_{n=0}^{\infty}$, $N, M \in \mathbb{N}$, are generated by a given optimization algorithm when applied to (P_{NM}), then $x^n_{NM} \in X$ for all $N, M \in \mathbb{N}$ and $n = 0, 1, 2, ...$* □

Based on the definition of X in Section III.B.3a, many optimization algorithms satisfy Assumption IV.2. We use the notation f^* to denote the optimal value of (P) and define the total error as

$$|f(x^n_{NM}) - f^*|, \tag{IV.6}$$

which is a measure of the quality of the solution obtained after applying n iterations of $\mathcal{A}$ to (P_{NM}). Based on Assumptions IV.1 and IV.2,

$$\begin{aligned} |f(x_{NM}^n) - f^*| &= |f(x_{NM}^n) - f_{NM}(x_{NM}^n) + f_{NM}(x_{NM}^n) - f_{NM}^* - f^* + f_{NM}^*| \\ &\leq \frac{K}{N^p} + \frac{K}{M^q} + \Delta_{NM}^n(\mathcal{A}), \end{aligned} \tag{IV.7}$$

where $\Delta_{NM}^n(\mathcal{A})$ is an upper bound on the optimization error after n iterations of optimization algorithm $\mathcal{A}$ are applied to (P_{NM}). In this chapter we consider several different expressions for $\Delta_{NM}^n(\mathcal{A})$ under various assumptions about the optimization algorithm, and hence also about (P_{NM}). Because it appears difficult to quantify the rate of convergence of the total error, as in Royset and Pee (2011) we concentrate on the rate of convergence of its upper bound in (IV.7). From (IV.7), it is clear that the rate of convergence of that bound gives a guaranteed minimum rate of convergence of the total error.

Based on (IV.7), it is evident that different choices of N, M, and n may result in different bounds on the total error. Let $b \in \mathbb{N}$ be the computational budget available to apply n iterations of $\mathcal{A}$ to (P_{NM}). To make it clear that the choice of N, M, and n would typically depend on b, we write N_b, M_b, and n_b. We refer to $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$, with $n_b, N_b, M_b \in \mathbb{N}$ for all $b \in \mathbb{N}$, as a discretization policy. A discretization policy specifies the level of discretization of (P) as well as the number of iterations of the optimization algorithm to complete for any computational budget. If $n_b, N_b, M_b \to \infty$, as $b \to \infty$, then based on Assumption IV.1, the bound on the discretization error vanishes. If we assume that a convergent optimization algorithm is applied to solve (P_{NM}), then the optimization error and, hence, the corresponding bound could vanish as well. For a given optimization algorithm $\mathcal{A}$, and $n, N, M \in \mathbb{N}$, we define the total error bound, denoted by $e(\mathcal{A}, n, N, M)$, as the right-hand side of (IV.7) and have

$$e(\mathcal{A}, n, N, M) \triangleq \frac{K}{N^p} + \frac{K}{M^q} + \Delta_{NM}^n(\mathcal{A}). \tag{IV.8}$$

In this chapter, we investigate the rate at which the total error bound $e(\mathcal{A}, n_b, N_b, M_b)$ vanishes as b tends to infinity for different discretization policies $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ and optimization algorithms $\mathcal{A}$. We give optimal discretization policies, which as discussed in detail below, attain the highest possible rate of convergence of the total error bound as the computational budget tends to infinity for a given class of optimization algorithms.

The analysis that follows relies on the following assumption about the computational work needed by an optimization algorithm to complete n iterations on (P_{NM}).

Assumption IV.3. *There exist a $\sigma = \sigma(\mathcal{A}) \in (0, \infty)$, a $\mu = \mu(\mathcal{A}) \in (0, \infty)$, and a $\nu = \nu(\mathcal{A}) \in (0, \infty)$ such that the computational work required by a given optimization algorithm $\mathcal{A}$ to complete $n \in \mathbb{N}$ iterations on (P_{NM}), $N, M \in \mathbb{N}$, is no larger than $\sigma n N^{\mu} M^{\nu}$.* □

Assumption IV.3 holds with $\mu = 1$ and $\nu = 2$ if the optimization algorithm $\mathcal{A}$ is a steepest descent method. This is because each iteration of these algorithms requires the calculation of $f_{NM}(x)$ at the current iterate $x \in X_N$, as well as the evaluation of the gradient $\nabla f_{NM}(x)$, which dominates the computational budget. In order to evaluate the gradient, using problem (GTP^c_{NM}) from Chapter III as an example, we see from (III.60) that $O(N)$ operations are required to find the physical states, while (III.61) and (III.73), respectively, indicate that $O(NM^2)$ operations are required to find the information state and adjoint. This leads to an overall computational complexity of $O(NM^2)$ to evaluate $\nabla f_{NM}(x)$.

The SQP algorithm in the TOMLAB SNOPT solver (see Gill et al., 2007) also seems to empirically follow Assumption IV.3 with $\mu = 1$ and $\nu = 2$ when used to solve (GTP^c) with $K = 1$ and $L = 10$ using Algorithm V.2, which will be discussed further in Chapter V, with $\bar{\eta}_0 = (\pi/4, 0 \in \mathbb{R}^{N_0})$, target and HVU parameters values given in Table 6 (except $v_{max} = 60$), and searcher and detection rate parameters given in the $k = 1$ columns of Tables 7 and 8, respectively. Table 1 shows the computational time in seconds required to complete five iterations of the SQP algorithm for various values

of N and M. Table 1 also shows the fitted values for a zero intercept linear regression done in R, version 2.13.0 (see Team, 2010), for the linear model $Y = aNM^2$. Figure 2 illustrates the close agreement between the actual and fitted values given in Table 1.

Table 1. Actual and fitted computational time in seconds for five iterations of the SQP algorithm in the TOMLAB SNOPT solver.

Instance	N	M	Actual	Fitted
1	16	7	1.854	1.427
2	16	9	2.164	2.359
3	32	7	2.620	2.855
4	32	9	4.176	4.719
5	64	7	4.532	5.709
6	64	9	8.655	9.438
7	128	7	10.152	11.419
8	64	11	13.337	14.099
9	128	9	15.279	18.876
10	128	11	24.154	28.197
11	128	15	42.168	52.433
12	128	17	60.439	67.348
13	256	11	69.144	56.395
14	256	15	102.924	104.866
15	256	17	120.854	134.695
16	512	15	196.666	209.733
17	512	17	289.129	269.390

In some implementations, particularly those involving discretization of optimal control problems using collocation methods, increasing the time discretization parameter, N, gives rise to a large number of equality constraints. This large number of equality constraints provides special structure that allows for sparse implementations where the computational work may grow slowly as N increases. Assumption IV.3 may account for the slow growth of the computational work in N, by selecting a value for $\mu \in (0, 1)$.

Based on Assumption IV.3, as in Royset and Pee (2011), we refer to a discretization policy $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ as asymptotically admissible if $\sigma n_b N_b^{\mu} M_b^{\nu} / b \to 1$,

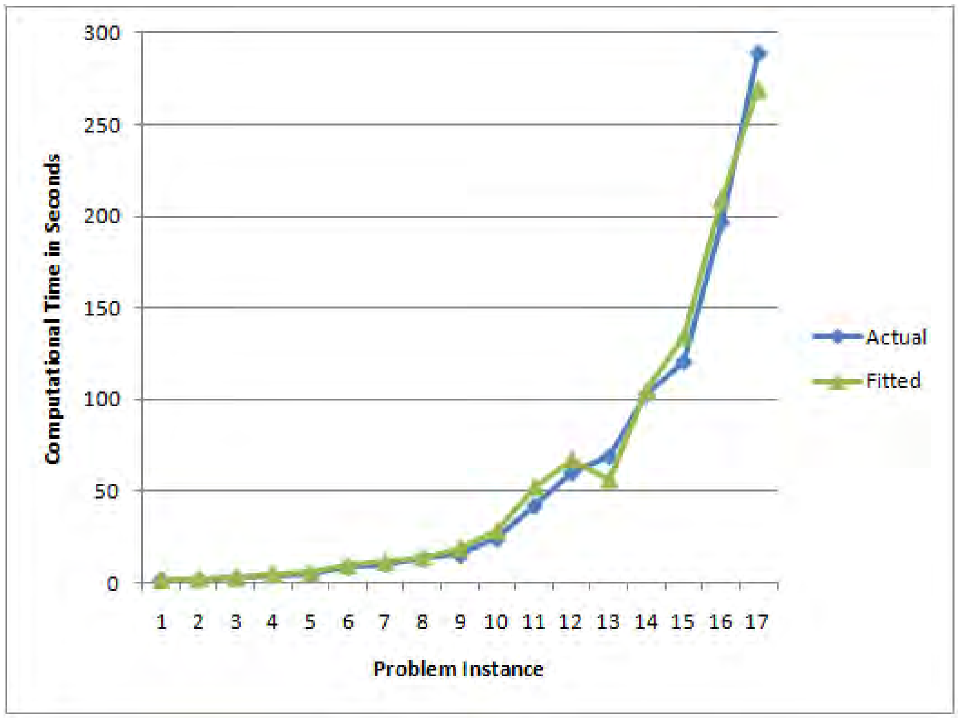

Figure 2. Regression fit for $Y = aNM^2$ model.

as $b \to \infty$. It is obvious that an asymptotically admissible discretization policy satisfies the computational budget in the limit as b tends to infinity. In the next section, we determine optimal asymptotically admissible discretization policies and corresponding rates of convergence of the total error bound under different assumptions about the optimization algorithm and, therefore, the optimization error bound $\Delta_{NM}^n(\mathcal{A})$.

C. RATE ANALYSIS FOR CLASSES OF ALGORITHMS

From (IV.7) we see that the total error bound consists of discretization and optimization error bounds. The discretization error bound depends on the discretization levels N and M, but not on the optimization algorithm used. The optimization error bound depends on the rate of convergence of the optimization algorithm used to solve (P_{NM}). In this section, we consider four cases: First, we assume that the optimization algorithm solves (P_{NM}) in a finite number of iterations. Second, we

investigate optimization algorithms with a superlinear rate of convergence towards an optimal solution of (P_{NM}). Third, we consider linearly convergent optimization algorithms. Fourth, we deal with sublinearly convergent algorithms.

1. Finite Optimization Algorithm

Suppose that the optimization algorithm used to solve (P_{NM}) is guaranteed to find an optimal solution in a finite number of iterations, independently of N and M. We define finitely convergent algorithms based on the following definition adapted from Royset and Pee (2011). We note that our definitions for superlinearly, linearly, and sublinearly convergent algorithms are also adapted from Royset and Pee (2011).

Definition IV.1. An optimization algorithm $\mathcal{A}$ converges finitely on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$ when X_{NM}^* is nonempty for $N \geq \bar{N}$, $M \geq \bar{M}$, and there exists a constant $\bar{n} \in \mathbb{N}$ such that for all $N \geq \bar{N}$, $M \geq \bar{M}$, $N, M \in \mathbb{N}$, a sequence $\{x_{NM}^n\}_{n=0}^{\infty}$ generated by $\mathcal{A}$ when applied to (P_{NM}) satisfies $x_{NM}^n \in X_{NM}^*$ for all $n \geq \bar{n}$. $\square$

No optimization algorithm converges finitely on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$ without strong structural assumptions on X_N and $f_{NM}(\cdot)$, such as linearity. In this dissertation, we are not concerned with instances of (P_{NM}) that are linear programs, which may allow for finite convergence. We include this case here to serve as an "ideal" case. It will be shown below that this case gives an upper bound on the rate of convergence of the total error bound using any optimization algorithm. Based on Definition IV.1, a finitely convergent optimization algorithm $\mathcal{A}^{\text{finite}}$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$ has no optimization error after performing a large enough number of iterations. We define $\Delta_{NM}^n(\mathcal{A}^{\text{finite}}) \triangleq 0$ and $e(\mathcal{A}^{\text{finite}}, n, N, M) \triangleq \frac{K}{N^p} + \frac{K}{M^q}$ for $n \geq \bar{n}$, $N \geq \bar{N}$, and $M \geq \bar{M}$, where K is as in Assumption IV.1 and $\bar{n}$, $\bar{N}$, and $\bar{M}$ are as in Definition IV.1. The next theorem gives the rate of convergence of the total error bound for this case.

Theorem IV.4. *Suppose that Assumption IV.1 holds and that $\mathcal{A}^{\text{finite}}$ is a finitely convergent algorithm on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, with $\bar{N}$ and $\bar{M}$ as in Assumption IV.1 and number of required iterations $\bar{n}$ as in Definition IV.1. Suppose also that $\mathcal{A}^{\text{finite}}$ satisfies Assumptions IV.2 and IV.3. If $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ is an asymptotically admissible discretization policy with $n_b = \bar{n}$ for all $b \in \mathbb{N}$, then*

$$\liminf_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} \geq -\frac{1}{(\mu/p + \nu/q)}, \tag{IV.9}$$

where p and q are as in Assumption IV.1, and μ and ν are as in Assumption IV.3.

Furthermore, if $N_b/b^{1/(\mu+p\nu/q)} \to a_1 \in (0,\infty)$ as $b \to \infty$, then

$$\liminf_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} = -\frac{1}{(\mu/p + \nu/q)}. \tag{IV.10}$$

Proof. For sufficiently large b, $\Delta^n_{NM}(\mathcal{A}^{\text{finite}}) = 0$ and $e(\mathcal{A}^{\text{finite}}, n, N, M) = \frac{K}{N^p} + \frac{K}{M^q}$, where K is as in Assumption IV.1. Consequently, for sufficiently large b,

$$\begin{aligned} \log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b) &= \log\left(\frac{K}{N_b^p} + \frac{K}{M_b^q}\right) && \text{(IV.11)} \\ &\geq \log\left(\max\left\{\frac{K}{N_b^p}, \frac{K}{M_b^q}\right\}\right) \\ &= \max\left\{\log\frac{K}{N_b^p}, \log\frac{K}{M_b^q}\right\}. && \text{(IV.12)} \end{aligned}$$

Hence,

$$\frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} \geq \max\left\{\frac{-p\log N_b + \log K}{\log b}, \frac{-q\log M_b + \log K}{\log b}\right\}. \tag{IV.13}$$

We now consider two cases, one for both of the terms inside the max function of (IV.13) when it is greater than or equal to the other term inside the max function.

Initially, we want the first term to be greater than or equal to the second term. This will be true if

$$-p\log N_b \geq -q\log M_b. \tag{IV.14}$$

We note that (IV.14) implies

$$M_b^\nu \geq N_b^{p\nu/q}. \tag{IV.15}$$

If we then use the bound on M_b^ν obtained from (IV.15), we have that

$$\begin{aligned}
\frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} &\geq \frac{-p\log N_b + \log K}{\log b} \\
&= \frac{-p\log N_b + \log K}{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right) + \log\left(\sigma\bar{n}N_b^\mu M_b^\nu\right)} \\
&\geq \frac{-p\log N_b + \log K}{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right) + \log(\sigma\bar{n}) + \log\left(N_b^{\frac{p\nu+q\mu}{q}}\right)} \\
&= \frac{-p + \frac{\log K}{\log N_b}}{\frac{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right)}{\log N_b} + \frac{\log(\sigma\bar{n})}{\log N_b} + \frac{p\nu+q\mu}{q}}. \qquad \text{(IV.16)}
\end{aligned}$$

Next, we consider the case where the second term inside the max function of (IV.13) is greater than or equal to the first term inside the max function. This will be true if

$$-q\log M_b \geq -p\log N_b. \qquad \text{(IV.17)}$$

We note that (IV.17) implies

$$N_b^\mu \geq M_b^{q\mu/p}. \qquad \text{(IV.18)}$$

If we then use the bound on N_b obtained from (IV.18), we have

$$\begin{aligned}
\frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} &\geq \frac{-q\log M_b + \log K}{\log b} \\
&= \frac{-q\log M_b + \log K}{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right) + \log\left(\sigma\bar{n}N_b^\mu M_b^\nu\right)} \\
&\geq \frac{-q\log M_b + \log K}{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right) + \log(\sigma\bar{n}) + \log\left(M_b^{\frac{p\nu+q\mu}{p}}\right)} \\
&= \frac{-q + \frac{\log K}{\log M_b}}{\frac{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right)}{\log M_b} + \frac{\log(\sigma\bar{n})}{\log M_b} + \frac{p\nu+q\mu}{p}}. \qquad \text{(IV.19)}
\end{aligned}$$

We now consider

$$\frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b}$$

as $b \to \infty$. Since $\sigma\bar{n}N_b^\mu M_b^\nu/b \to 1$ as $b \to \infty$, we consider two cases: one where at least one of the parameters N_b or M_b does not increase without bound as $b \to \infty$, and another where $N_b \to \infty$ and $M_b \to \infty$ as $b \to \infty$. For the first case, at least one of the parameters N_b or M_b remains finite as $b \to \infty$. Then based on (IV.11), $e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b) \in \left(0, \limsup_{b\to\infty} \frac{K}{N_b^p} + \frac{K}{M_b^q}\right)$, and we have that

$$\liminf_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} \geq 0 > -\frac{1}{(\mu/p + \nu/q)}. \tag{IV.20}$$

For the second case, where both of the parameters N_b and M_b increase without bound as $b \to \infty$, we have that

$$\begin{aligned}
&\liminf_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)}{\log b} \\
\geq\ & \min\left\{ \liminf_{b\to\infty} \frac{-p + \frac{\log K}{\log N_b}}{\frac{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right)}{\log N_b} + \frac{\log(\sigma\bar{n})}{\log N_b} + \frac{p\nu+q\mu}{q}}, \right. \\
& \left. \liminf_{b\to\infty} \frac{-q + \frac{\log K}{\log M_b}}{\frac{\log\left(\frac{b}{\sigma\bar{n}N_b^\mu M_b^\nu}\right)}{\log M_b} + \frac{\log(\sigma\bar{n})}{\log M_b} + \frac{p\nu+q\mu}{p}} \right\} \\
=\ & \min\left\{ -\frac{1}{(\mu/p + \nu/q)}, -\frac{1}{(\mu/p + \nu/q)} \right\} = -\frac{1}{(\mu/p + \nu/q)},
\end{aligned}$$

which completes the proof for the first part of the theorem.

Next, let $\{(n_b, N_b, M_b)\}_{b=1}^\infty$ be an asymptotically admissible discretization policy with $n_b = \bar{n}$ satisfying $N_b/b^{1/(\mu+p\nu/q)} \to a_1 \in (0, \infty)$ as $b \to \infty$. For notational simplification we define

$$B_1 = \frac{Kb^{\frac{1}{\mu/p+\nu/q}}}{N^p}, \tag{IV.21}$$

Then, for sufficiently large b, $e(\mathcal{A}^{\text{finite}}, n, N, M) = \frac{K}{N^p} + \frac{K}{M^q}$, and algebraic manipulation gives

$$\begin{aligned} & e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b) \\ = \;& B_1 b^{\frac{-1}{\mu/p+\nu/q}} + \frac{K}{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}} \frac{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}{M^q} \\ = \;& b^{\frac{-1}{\mu/p+\nu/q}} \left[B_1 + \frac{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}{M^q} \frac{K \sigma^{q/\nu} n^{q/\nu} N^{\mu q/\nu}}{b^{q/\nu}} b^{\frac{1}{\mu/p+\nu/q}} \right]. \end{aligned} \tag{IV.22}$$

Algebraic manipulation can be used to show that

$$b^{\frac{q\mu}{\mu\nu+p\nu^2/q}} b^{-q/\nu} b^{\frac{1}{\mu/p+\nu/q}} = b^0 = 1. \tag{IV.23}$$

Then, since $\sigma n_b N_b^\mu M_b^\nu / b \to 1$, the sum of terms in brackets in (IV.22), with n, N, and M replaced by $\bar{n}$, N_b, and M_b, respectively, tends to a constant as $b \to \infty$. The conclusion for the second part of the theorem then follows from (IV.22) after taking logarithms, dividing by $\log b$, and taking limits. □

Theorem IV.4 shows that $e(\mathcal{A}^{\text{finite}}, n_b, N_b, M_b)$ converges at a rate $b^{-1/(\mu/p+\nu/q)}$ under the stated discretization policy. From (IV.8) we see that the total error bound includes the discretization error bound. This means the total error bound cannot converge faster than the rate $b^{-1/(\mu/p+\nu/q)}$ no matter which optimization algorithm is used.

It would be difficult to implement the asymptotically admissible discretization policy stated in Theorem IV.4 because $\bar{n}$ may be unknown. Despite this difficulty, the rate of convergence obtained in Theorem IV.4 is useful as an upper bound on the rate that can be achieved by any optimization algorithm, and will be used as a benchmark for comparison in the case of superlinear, linear, and sublinear algorithms below.

2. Superlinear Optimization Algorithm

We will now consider superlinearly convergent optimization algorithms. We define superlinearly convergent algorithms based on the following definition.

Definition IV.2. An optimization algorithm $\mathcal{A}$ converges superlinearly with order $\gamma \in (1,\infty)$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$ when X_N^* is nonempty for $N \geq \bar{N}$, $M \geq \bar{M}$, and there exist constants $\bar{n} \in \mathbb{N}$, $\bar{c} \in [0,\infty)$, and $\rho \in [0,1)$ such that $\bar{c}^{1/(\gamma-1)}(f_{NM}(x_{NM}^n) - f_{NM}^*) \leq \rho$ and

$$\frac{f_{NM}\left(x_{NM}^{n+1}\right) - f_{NM}^*}{\left[f_{NM}\left(x_{NM}^n\right) - f_{NM}^*\right]^\gamma} \leq \bar{c}, \tag{IV.24}$$

for all $n \geq \bar{n}$, $n \in \mathbb{N}$, $N \geq \bar{N}$, and $M \geq \bar{M}$, $N, M \in \mathbb{N}$. □

Definition IV.2 requires that the optimization algorithm achieve a superlinear rate of convergence for sufficiently large n. This is usually the case with $\gamma = 2$ for Newtonian methods applied to strongly convex instances of (P_{NM}) with twice Lipschitz continuously differentiable objective functions. The next lemma gives a total error bound for a superlinearly convergent algorithm.

Lemma IV.5. *Suppose that Assumption IV.1 holds and that $\mathcal{A}^{\text{super}}$ is a superlinearly convergent algorithm with order $\gamma \in (1,\infty)$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, with $\bar{N}$ and $\bar{M}$ as in Assumption IV.1. Let $\{x_{NM}^n\}_{n=0}^\infty$ be the iterates generated by $\mathcal{A}^{\text{super}}$ when applied to (P_{NM}), $N \in \mathbb{N}$, $N \geq \bar{N}$, $M \in \mathbb{N}$, $M \geq \bar{M}$. Suppose also that $\mathcal{A}^{\text{super}}$ satisfies Assumptions IV.2 and IV.3. Then, there exist constants $c \in (0,1)$, $\kappa \in [0,\infty)$, and $\bar{n} \in \mathbb{N}$ such that*

$$f(x_{NM}^n) - f^* \leq c^{\gamma^n}\kappa + \frac{K}{N^p} + \frac{K}{M^q}, \tag{IV.25}$$

for all $n \geq \bar{n}$, $n \in \mathbb{N}$, $N \geq \bar{N}$, $N \in \mathbb{N}$, and $M \geq \bar{M}$, $M \in \mathbb{N}$, where K, p, and q are as in Assumption IV.1.

Proof. The proof follows the same arguments as those for the proof of Lemma 1 in Royset & Pee (2011). □

From Lemma IV.5, we adopt the upper bound on the optimization error

$$\Delta_{NM}^n(\mathcal{A}^{\text{super}}) \triangleq c^{\gamma^n}\kappa \tag{IV.26}$$

for a superlinearly convergent optimization algorithm $\mathcal{A}^{\text{super}}$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, where c and κ are as in Lemma IV.5. Then for $n, N, M \in \mathbb{N}$, we define the total error bound

$$e(\mathcal{A}^{\text{super}}, n, N, M) \triangleq c^{\gamma^n}\kappa + \frac{K}{N^p} + \frac{K}{M^q}. \tag{IV.27}$$

The next result shows that if we select a particular discretization policy, then a superlinearly convergent optimization algorithm results in the same rate of convergence of the total error bound as a finitely convergent algorithm. Therefore, the policy specified in the following theorem is optimal in the sense that no other policy guarantees a better rate of convergence.

Theorem IV.6. *Suppose that Assumption IV.1 holds and that $\mathcal{A}^{\text{super}}$ is a superlinearly convergent algorithm of order $\gamma \in (1,\infty)$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, with $\bar{N}$ and $\bar{M}$ as in Assumption IV.1. Suppose also that $\mathcal{A}^{\text{super}}$ satisfies Assumptions IV.2 and IV.3. If $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ is an asymptotically admissible discretization policy with*

$$\frac{n_b \log \gamma}{\log\left(\frac{\log b}{-\log c}\right)} \to a_1 = 1 \text{ and } \frac{pq}{\mu q + \nu p} < 1, \tag{IV.28}$$

or

$$\frac{n_b \log \gamma}{\log\left(\frac{\log b}{-\log c}\right)} \to a_1 \in (1,\infty), \tag{IV.29}$$

and

$$\frac{N_b}{\left[\frac{b \log \gamma}{\sigma \log\left(\frac{\log b}{-\log c}\right)}\right]^{\frac{q}{\mu q + \nu p}}} \to a_2 \in (0,\infty), \tag{IV.30}$$

then

$$\lim_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{super}}, n_b, N_b, M_b)}{\log b} = -\frac{1}{(\mu/p + \nu/q)}, \tag{IV.31}$$

where p and q are as in Assumption IV.1, and μ and ν are as in Assumption IV.3.

Proof. For notational simplification we define

$$B_1 = \exp\left(\log \kappa + \log c \left(\frac{\log b}{-\log c}\right)^{\frac{n \log \gamma}{\log\left(\frac{\log b}{-\log c}\right)}}\right), \tag{IV.32}$$

$$B_2 = \frac{K\left(\frac{b \log \gamma}{\sigma \log\left(\frac{\log b}{-\log c}\right)}\right)^{\frac{pq}{\mu q + \nu p}}}{N^p}, \tag{IV.33}$$

and

$$B_3 = \left(\frac{b \log \gamma}{\sigma \log\left(\frac{\log b}{-\log c}\right)}\right)^{\frac{-pq}{\mu q + \nu p}}. \tag{IV.34}$$

Algebraic manipulation of (IV.27) gives

$$\begin{aligned}
& e(\mathcal{A}^{\text{super}}, n_b, N_b, M_b) \\
= \ & \exp\left(\log\kappa + \gamma^{\frac{n\log\gamma}{\log\left(\frac{\log b}{-\log c}\right)}\log_\gamma\left(\frac{\log b}{-\log c}\right)}\log c\right) + B_2B_3 + \frac{K}{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}\frac{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}{M^q} \\
= \ & B_1 + B_2B_3 + \frac{K}{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}\frac{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}{M^q} \\
= \ & B_3\left[\frac{B_1}{B_3} + B_2 + \frac{\frac{\left(\frac{b}{\sigma n N^\mu}\right)^{q/\nu}}{M^q}\frac{K\sigma^{q/\nu}n^{q/\nu}N^{\mu q/\nu}}{b^{q/\nu}}}{B_3}\right].
\end{aligned} \tag{IV.35}$$

We now focus on the following portion of the first term inside the square brackets in (IV.35),

$$B_1 b^{\frac{pq}{\mu q+\nu p}} = \kappa\exp\left(\log c\left(\frac{\log b}{-\log c}\right)^{\frac{n\log\gamma}{\log\left(\frac{\log b}{-\log c}\right)}}\right)b^{\frac{pq}{\mu q+\nu p}} \tag{IV.36}$$

To simplify the analysis, we take the logarithm and obtain

$$\begin{aligned}
& \log\kappa + \log c\left(\frac{\log b}{-\log c}\right)^{\frac{n\log\gamma}{\log\left(\frac{\log b}{-\log c}\right)}} + \frac{pq}{\mu q+\nu p}\log b \\
= \ & \log\kappa + \log b\left[\frac{\log c}{(-\log c)^{\frac{n\log\gamma}{\log\left(\frac{\log b}{-\log c}\right)}}}(\log b)^{\frac{n\log\gamma}{\log\left(\frac{\log b}{-\log c}\right)}-1} + \frac{pq}{\mu q+\nu p}\right].
\end{aligned} \tag{IV.37}$$

Since $\sigma n_b N_b^\mu M_b^\nu / b \to 1$,

$$\begin{aligned}
\frac{n_b\log\gamma}{\log\left(\frac{\log b}{-\log c}\right)} &\to a_1, \\
\frac{N_b}{\left[\frac{b\log\gamma}{\sigma\log\left(\frac{\log b}{-\log c}\right)}\right]^{\frac{q}{\mu q+\nu p}}} &\to a_2,
\end{aligned}$$

and, due to the facts that $\log c < 0$ and $a_1 = 1$ and $\frac{pq}{\mu q+\nu p} < 1$ or $a_1 \in (1,\infty)$, the expression in (IV.37) goes to $-\infty$, as $b\to\infty$, we see that the sum of terms in brackets in (IV.35), with n, N, and M replaced by n_b, N_b, and M_b, respectively, tends to a constant as $b\to\infty$. The conclusion then follows from (IV.35) after taking logarithms, dividing by $\log b$, and taking limits. □

While no discretization policy can guarantee a rate of convergence better than $b^{-1/(\mu/p+\nu/q)}$, it is possible to do worse. If, for example, $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ is an asymptotically admissible discretization policy with

$$\frac{n_b}{b^{1/2}} \to a_1 = 1 \text{ and } \frac{pq}{\mu q + \nu p} < 1, \tag{IV.38}$$

or

$$\frac{n_b}{b^{1/2}} \to a_1 \in (1, \infty), \tag{IV.39}$$

and

$$\frac{N_b}{\left[\frac{b^{1/2}}{\sigma}\right]^{\frac{q}{\mu q+\nu p}}} \to a_2 \in (0, \infty), \tag{IV.40}$$

then it can be shown using arguments similar to those used in the proof of Theorem IV.6 that

$$\lim_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{super}}, n_b, N_b, M_b)}{\log b} = -\frac{1}{\left(\frac{2\mu}{p} + \frac{2\nu}{q}\right)}, \tag{IV.41}$$

which is worse than the optimal rate by a factor of two.

3. Linear Optimization Algorithm

We now consider linearly convergent optimization algorithms. We define linearly convergent algorithms based on the following definition.

Definition IV.3. An optimization algorithm $\mathcal{A}$ converges linearly on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$ when X_N^* is nonempty for $N \geq \bar{N}$, $M \geq \bar{M}$ and there exist constants $\bar{n} \in \mathbb{N}$ and $\bar{c} \in (0, 1)$ such that

$$\frac{f_{NM}\left(x_{NM}^{n+1}\right) - f_{NM}^*}{f_{NM}\left(x_{NM}^{n}\right) - f_{NM}^*} \leq \bar{c}, \tag{IV.42}$$

for all $n \geq \bar{n}$, $n \in \mathbb{N}$, $N \geq \bar{N}$, $N \in \mathbb{N}$, and $M \geq \bar{M}$, $M \in \mathbb{N}$. □

Definition IV.3 slightly extends a standard definition of linear convergence to require that the rate of convergence coefficient holds for all N and M sufficiently large. This is satisfied by many gradient methods, such as the steepest descent method and projected gradient method, when applied to (P_{NM}) under the assumption that the

objective function is strongly convex and twice continuously differentiable and that X is convex. The next lemma gives a total error bound for a linearly convergent algorithm.

Lemma IV.7. *Suppose that Assumption IV.1 holds and that $\mathcal{A}^{\text{linear}}$ is a linearly convergent algorithm on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, with $\bar{N}$ and $\bar{M}$ as in Assumption IV.1. Let $\{x_{NM}^n\}_{n=0}^{\infty}$ be the iterates generated by $\mathcal{A}^{\text{linear}}$ when applied to (P_{NM}), $N \in \mathbb{N}$, $N \geq \bar{N}$, $M \in \mathbb{N}$, $M \geq \bar{M}$. Suppose also that there exists a constant $C \in \mathbb{R}$ such that $f_{NM}(x_N^n) \leq C$ for all $n \in \mathbb{N}$, $N \geq \bar{N}$, $N \in \mathbb{N}$, and $M \geq \bar{M}$, $M \in \mathbb{N}$, and that $\mathcal{A}^{\text{linear}}$ satisfies Assumption IV.2. Then, there exists a constant $\kappa \in [0, \infty)$ such that*

$$f(x_{NM}^n) - f^* \leq \bar{c}^n \kappa + \frac{K}{N^p} + \frac{K}{M^q}, \tag{IV.43}$$

for all $n \geq \bar{n}$, $N \geq \bar{N}$, and $M \geq \bar{M}$, where $\bar{c}$ and $\bar{n}$ are as in Definition IV.3, and K, p, and q are as in Assumption IV.1.

Proof. Based on Assumption IV.1 and the fact that $\mathcal{A}^{\text{linear}}$ is linearly convergent, we obtain that

$$\begin{aligned} |f(x_{NM}^n) - f^*| &\leq f_{NM}(x_{NM}^n) + \frac{K}{N^p} + \frac{K}{M^q} - f_{NM}^* \\ &\leq \bar{c}^{n-\bar{n}} \left[f_{NM}(x_{NM}^{\bar{n}}) - f_{NM}^* \right] + \frac{K}{N^p} + \frac{K}{M^q} \\ &\leq \bar{c}^n \left(\bar{c}^{-\bar{n}} \left(C - f^* + \frac{K}{\bar{N}^p} + \frac{K}{\bar{M}^q} \right) \right) + \frac{K}{N^p} + \frac{K}{M^q}. \end{aligned} \tag{IV.44}$$

Hence, the results hold with $\kappa = \bar{c}^{-\bar{n}} \left(C - f^* + \frac{K}{\bar{N}^p} + \frac{K}{\bar{M}^q} \right)$. □

From Lemma IV.7, we adopt the upper bound on the optimization error

$$\Delta_{NM}^n(\mathcal{A}^{\text{linear}}) \triangleq \bar{c}^n \kappa, \tag{IV.45}$$

for a linearly convergent optimization algorithm $\mathcal{A}^{\text{linear}}$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, where $\bar{c}$ and κ are as in Lemma IV.7. Then for $n, N, M \in \mathbb{N}$, we define the total error bound

$$e(\mathcal{A}^{\text{linear}}, n, N, M) \triangleq \bar{c}^n \kappa + \frac{K}{N^p} + \frac{K}{M^q}. \tag{IV.46}$$

The next result shows that if we select a particular discretization policy, then a linearly convergent optimization algorithm can also attain the best possible rate of convergence of the total error bound given in Theorems IV.4 and IV.6.

Theorem IV.8. *Suppose that $\mathcal{A}^{\text{linear}}$ satisfies the assumptions of Lemma IV.7 and, in addition, Assumption IV.3 holds. If $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ is an asymptotically admissible discretization policy with*

$$\frac{-n_b \log \bar{c}}{\log b} \to a_1 \in \left[\frac{pq}{\mu q + \nu p}, \infty\right), \tag{IV.47}$$

and

$$\frac{N_b}{\left[\frac{-b \log \bar{c}}{\sigma \log b}\right]^{\frac{q}{\mu q + \nu p}}} \to a_2 \in (0, \infty), \tag{IV.48}$$

where $\bar{c}$ is as in Definition IV.3, p and q are as in Assumption IV.1, and μ and ν are as in Assumption IV.3, then

$$\lim_{b \to \infty} \frac{\log e(\mathcal{A}^{\text{linear}}, n_b, N_b, M_b)}{\log b} = -\frac{1}{\mu/p + \nu/q}. \tag{IV.49}$$

Proof. For notational simplification we define

$$B_1 = \frac{K \left[\frac{-b \log \bar{c}}{\sigma \log b}\right]^{\frac{pq}{\mu q + \nu p}}}{N^p}, \tag{IV.50}$$

and

$$B_2 = \left(\frac{-b \log \bar{c}}{\sigma \log b}\right)^{\frac{-pq}{\mu q + \nu p}}. \tag{IV.51}$$

Algebraic manipulation of (IV.46) gives

$$\begin{aligned}
& e(\mathcal{A}^{\text{linear}}, n_b, N_b, M_b) \\
= \;& \exp\left(\log \kappa + \frac{-n \log \bar{c}}{\log b} \frac{\log b}{-\log \bar{c}} \log \bar{c}\right) + B_1 B_2 + \frac{K}{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}} \frac{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}{M^q} \\
= \;& \exp\left(\log \kappa + \frac{n \log \bar{c}}{\log b} \log b\right) + B_1 B_2 + \frac{K}{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}} \frac{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}{M^q} \\
= \;& \frac{\kappa}{b^{\frac{-n \log \bar{c}}{\log b}}} + B_1 B_2 + \frac{K}{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}} \frac{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}{M^q} \\
= \;& B_2 \left[\frac{\kappa}{B_2 b^{\frac{-n \log \bar{c}}{\log b}}} + B_1 + \frac{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}{B_2 M^q} \frac{K \sigma^{q/\nu} n^{q/\nu} N^{\mu q/\nu}}{b^{q/\nu}}\right].
\end{aligned} \tag{IV.52}$$

Since

$$\frac{-n_b \log \bar{c}}{\log b} \to a_1 \in [\frac{pq}{\mu q + \nu p}, \infty),$$

the expression

$$\frac{\kappa}{B_2 b^{\frac{-n \log \bar{c}}{\log b}}} \tag{IV.53}$$

goes to 0, as $b \to \infty$. In addition, we know that $\sigma n_b N_b^\mu M_b^\nu / b \to 1$, and

$$\frac{N_b}{\left[\frac{-b \log \bar{c}}{\sigma \log b}\right]^{\frac{q}{\mu q + \nu p}}} \to a_2,$$

as $b \to \infty$. Consequently, the sum of terms in brackets in (IV.52), with n, N, and M replaced by n_b, N_b, and M_b, respectively, tends to a constant as $b \to \infty$. The conclusion then follows from (IV.52) after taking logarithms, dividing by $\log b$, and taking limits. □

4. Sublinear Rate of Convergence

Next we consider sublinearly convergent optimization algorithms. We define sublinearly convergent algorithms based on the following definition.

Definition IV.4. An optimization algorithm $\mathcal{A}$ converges sublinearly with degree $\gamma \in (0, \infty)$ on $\{(P_{NM})\}_{N=\bar{N}, M=\bar{M}}^{\infty,\infty}$ when X_N^* is nonempty for $N \geq \bar{N}$, $M \geq \bar{M}$ and there exists a constant $C \in [0, \infty)$ such that

$$f_{NM}\left(x_{NM}^n\right) - f_{NM}^* \leq \frac{C}{n^\gamma}, \tag{IV.54}$$

for all $n \in \mathbb{N}$, $N \geq \bar{N}$, $N \in \mathbb{N}$, and $M \geq \bar{M}$, $M \in \mathbb{N}$. □

The subgradient method is sublinearly convergent in the sense of Definition IV.4 with $\gamma = 1/2$ and $C = D_X L_f$ when (P_{NM}) is convex, where D_X is the diameter of X_N and L_f is a Lipschitz constant of $f_{NM}(\cdot)$ on X; see Nesterov (2004), pp. 142-143. Based on Definition IV.4, we define the optimization error bound for a sublinearly convergent optimization algorithm $\mathcal{A}^{\text{sublin}}$ by

$$\Delta_{NM}^n(\mathcal{A}^{\text{sublin}}) \triangleq \frac{C}{n^\gamma}, \tag{IV.55}$$

and the total error bound for $n, N, M \in \mathbb{N}$ by

$$e(\mathcal{A}^{\text{sublin}}, n, N, M) \triangleq \frac{C}{n^\gamma} + \frac{K}{N^p} + \frac{K}{M^q}. \tag{IV.56}$$

The next theorem gives an optimal discretization policy for a sublinearly convergent optimization algorithm as well as its corresponding rate of convergence of the total error bound.

Theorem IV.9. *Suppose that Assumption IV.1 holds and that $\mathcal{A}^{\text{sublin}}$ is a sublinearly convergent algorithm with degree $\gamma \in (0,\infty)$ on $\{(P_{NM})\}_{N=\bar{N},M=\bar{M}}^{\infty,\infty}$, with $\bar{N}$ and $\bar{M}$ as in Assumption IV.1. Suppose also that $\mathcal{A}^{\text{sublin}}$ satisfies Assumptions IV.2 and IV.3, and that $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ is an asymptotically admissible discretization policy. Then,*

$$\lim_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b)}{\log b} \geq -\frac{1}{\frac{1}{\gamma}+\frac{\mu}{p}+\frac{\nu}{q}}, \tag{IV.57}$$

where γ is as in Definition IV.4, p and q are as in Assumption IV.1, and μ and ν are as in Assumption IV.3.

Furthermore, if $n_b/b^{1/(\mu\gamma/p+\nu\gamma/q+1)} \to a_1 \in (0,\infty)$ and

$$\frac{N_b}{\left(\frac{b^{\frac{\mu\gamma/p+\nu\gamma/q}{\mu\gamma/p+\nu\gamma/q+1}}}{\sigma}\right)^{\frac{q}{\mu q+\nu p}}} \to a_2 \in (0,\infty), \tag{IV.58}$$

as $b\to\infty$, then

$$\lim_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b)}{\log b} = -\frac{1}{\frac{1}{\gamma}+\frac{\mu}{p}+\frac{\nu}{q}}. \tag{IV.59}$$

Proof. For any $n, N, M \in \mathbb{N}$,

$$\begin{aligned}
\log e(\mathcal{A}^{\text{sublin}}, n, N, M) &= \log\left(\frac{C}{n^\gamma}+\frac{K}{N^p}+\frac{K}{M^q}\right) && \text{(IV.60)}\\
&\geq \log\left(\max\left\{\frac{C}{n^\gamma},\frac{K}{N^p},\frac{K}{M^q}\right\}\right) &&\\
&= \max\left\{\log c-\gamma\log n, \log\frac{K}{N^p}, \log\frac{K}{M^q}\right\}. && \text{(IV.61)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\log e(\mathcal{A}^{\text{sublin}}, n, N, M)}{\log b}\\
\geq\ & \max\left\{\frac{-\gamma\log n+\log C}{\log b}, \frac{-p\log N+\log K}{\log b}, \frac{-q\log M+\log K}{\log b}\right\}. \quad \text{(IV.62)}
\end{aligned}$$

We now consider three cases, one for each of the three terms inside the max function of (IV.62) when it is greater than or equal to the other two terms inside the max function.

Initially, we want the first term to be the greater than or equal to the other two terms. This will be true if

$$-\gamma \log n_b \geq -p \log N_b \tag{IV.63}$$

and

$$-\gamma \log n_b \geq -q \log M_b. \tag{IV.64}$$

We note that (IV.63) implies

$$N_b^{\mu} \geq n_b^{\mu\gamma/p}, \tag{IV.65}$$

and (IV.64) implies

$$M_b^{\nu} \geq n_b^{\gamma\nu/q}. \tag{IV.66}$$

If we then use the bounds on N_b^{μ} and M_b^{ν} obtained from (IV.65) and (IV.66), respectively, we have that

$$\begin{aligned}
\frac{\log e(\mathcal{A}^{\text{sublin}}, n, N, M)}{\log b} &\geq \frac{-\gamma \log n_b + \log C}{\log b} \\
&= \frac{-\gamma \log n_b + \log C}{\log\left(\frac{b}{\sigma n_b N_b^{\mu} M_b^{\nu}}\right) + \log\left(\sigma n_b N_b^{\mu} M_b^{\nu}\right)} \\
&\geq \frac{-\gamma \log n_b + \log C}{\log\left(\frac{b}{\sigma n_b N_b^{\mu} M_b^{\nu}}\right) + \log \sigma + \log\left(n_b^{\frac{pq+q\mu\gamma+p\nu\gamma}{pq}}\right)} \\
&= \frac{-\gamma + \frac{\log C}{\log n_b}}{\frac{\log\left(\frac{b}{\sigma n_b N_b^{\mu} M_b^{\nu}}\right)}{\log n_b} + \frac{\log \sigma}{\log n_b} + \frac{pq+q\mu\gamma+p\nu\gamma}{pq}}.
\end{aligned} \tag{IV.67}$$

We now consider the case where the second term inside the max function of (IV.62) is greater than or equal to the other two terms. This will be true if

$$-p \log N_b \geq -\gamma \log n_b, \tag{IV.68}$$

and

$$-p \log N_b \geq -q \log M_b. \tag{IV.69}$$

We note that (IV.68) implies

$$n_b \geq N_b^{p/\gamma}, \tag{IV.70}$$

and (IV.69) implies

$$M_b^\nu \geq N_b^{p\nu/q}. \tag{IV.71}$$

If we then use the bounds on n_b and M_b^ν obtained from (IV.70) and (IV.71), respectively, we have that

$$\begin{aligned}
\frac{\log e(\mathcal{A}^{\text{sublin}}, n, N, M)}{\log b} &\geq \frac{-p\log N_b + \log K}{\log b} \\
&= \frac{-p\log N_b + \log K}{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right) + \log\left(\sigma n_b N_b^\mu M_b^\nu\right)} \\
&\geq \frac{-p\log N_b + \log K}{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right) + \log\sigma + \log\left(N_b^{\frac{pq+q\mu\gamma+p\nu\gamma}{q\gamma}}\right)} \\
&= \frac{-p + \frac{\log K}{\log N_b}}{\frac{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right)}{\log N_b} + \frac{\log\sigma}{\log N_b} + \frac{pq+q\mu\gamma+p\nu\gamma}{q\gamma}}. \qquad \text{(IV.72)}
\end{aligned}$$

Finally, we consider the case where the third term inside the max function of (IV.62) is greater than or equal to the other two terms. This will be true if

$$-q\log M_b \geq -\gamma\log n_b, \tag{IV.73}$$

and

$$-q\log M_b \geq -p\log N_b. \tag{IV.74}$$

We note that (IV.73) implies

$$n_b \geq M_b^{q/\gamma}, \tag{IV.75}$$

and (IV.74) implies

$$N_b^\mu \geq M_b^{q\mu/p}. \tag{IV.76}$$

If we then use the bounds on n_b and N_b^μ obtained from (IV.75) and (IV.76), respectively, we have that

$$\begin{aligned}
\frac{\log e(\mathcal{A}^{\text{sublin}}, n, N, M)}{\log b} &\geq \frac{-q\log M_b + \log K}{\log b} \\
&= \frac{-q\log M_b + \log K}{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right) + \log\left(\sigma n_b N_b^\mu M_b^\nu\right)} \\
&\geq \frac{-q\log M_b + \log K}{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right) + \log\sigma + \log\left(M_b^{\frac{pq+q\mu\gamma+p\nu\gamma}{p\gamma}}\right)} \\
&= \frac{-q + \frac{\log K}{\log M_b}}{\frac{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right)}{\log M_b} + \frac{\log\sigma}{\log M_b} + \frac{pq+q\mu\gamma+p\nu\gamma}{p\gamma}}. \qquad \text{(IV.77)}
\end{aligned}$$

We now consider

$$\frac{\log e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b)}{\log b},$$

as $b \to \infty$. Since $\sigma n_b N_b^\mu M_b^\nu / b \to 1$, as $b \to \infty$, we consider two cases: one where at least one of the parameters n_b, N_b, or M_b does not increase without bound as $b \to \infty$, and another where $n_b \to \infty$, $N_b \to \infty$ and $M_b \to \infty$, as $b \to \infty$. For the first case, at least one of the parameters n_b, N_b, or M_b remains finite as $b \to \infty$. Then based on (IV.60), $e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b) \in \left(0, \limsup_{b\to\infty} \frac{C}{n_b^\gamma} + \frac{K}{N_b^p} + \frac{K}{M_b^q}\right)$, and we have that

$$\liminf_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b)}{\log b} \geq 0 > -\frac{1}{\frac{1}{\gamma} + \frac{\mu}{p} + \frac{\nu}{q}}. \qquad \text{(IV.78)}$$

For the second case, where all three of the parameters n_b, N_b, and M_b increase without bound as $b \to \infty$, we have that

$$\liminf_{b\to\infty} \frac{\log e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b)}{\log b}$$

$$\geq \min\left\{ \liminf_{b\to\infty} \frac{-\gamma + \frac{\log C}{\log n_b}}{\frac{\log\left(\frac{b}{\sigma n_b N_b^\mu M_b^\nu}\right)}{\log n_b} + \frac{\log\sigma}{\log n_b} + \frac{pq+q\mu\gamma+p\nu\gamma}{pq}}, \right.$$

$$\left.\liminf_{b\to\infty}\frac{-p+\frac{\log K}{\log N_b}}{\frac{\log\left(\frac{b}{\sigma n_b N_b^{\mu} M_b^{\nu}}\right)}{\log N_b}+\frac{\log\sigma}{\log N_b}+\frac{pq+q\mu\gamma+p\nu\gamma}{q\gamma}},\liminf_{b\to\infty}\frac{-q+\frac{\log K}{\log M_b}}{\frac{\log\left(\frac{b}{\sigma n_b N_b^{\mu} M_b^{\nu}}\right)}{\log M_b}+\frac{\log\sigma}{\log M_b}+\frac{pq+q\mu\gamma+p\nu\gamma}{p\gamma}}\right\}$$

$$\begin{aligned}
&= \min\left\{-\frac{1}{\frac{1}{\gamma}+\frac{\mu}{p}+\frac{\nu}{q}},-\frac{1}{\frac{1}{\gamma}+\frac{\mu}{p}+\frac{\nu}{q}},-\frac{1}{\frac{1}{\gamma}+\frac{\mu}{p}+\frac{\nu}{q}}\right\}\\
&= -\frac{1}{\frac{1}{\gamma}+\frac{\mu}{p}+\frac{\nu}{q}}. \qquad \text{(IV.79)}
\end{aligned}$$

Which completes the proof for the first part of the theorem.

Next, let $\{(n_b, N_b, M_b)\}_{b=1}^{\infty}$ be an asymptotically admissible discretization policy satisfying $n_b/b^{1/(\mu\gamma/p+\nu\gamma/q+1)} \to a_1 \in (0,\infty)$ and

$$\frac{N_b}{\left(\frac{b^{\frac{\mu\gamma/p+\nu\gamma/q}{\mu\gamma/p+\nu\gamma/q+1}}}{\sigma}\right)^{\frac{q}{\mu q+\nu p}}} \to a_2 \in (0,\infty), \qquad \text{(IV.80)}$$

as $b\to\infty$. For notational simplification we define

$$B_1 = \frac{Cb^{\frac{\gamma}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}}}{n^{\gamma}}, \qquad \text{(IV.81)}$$

$$B_2 = \frac{K\left[\frac{b^{\frac{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}}}{\sigma}\right]^{\frac{pq}{\mu q+\nu p}}}{N^p}, \qquad \text{(IV.82)}$$

and

$$B_3 = \left[\frac{b^{\frac{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}}}{\sigma}\right]^{\frac{-pq}{\mu q+\nu p}}. \qquad \text{(IV.83)}$$

Then, algebraic manipulation of (IV.56) gives

$$\begin{aligned}
&e(\mathcal{A}^{\text{sublin}}, n_b, N_b, M_b)\\
&= B_1 b^{\frac{-\gamma}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}} + B_2B_3 + \frac{K}{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}\frac{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}{M^q}\\
&= B_3\left[\frac{B_1}{B_3}b^{\frac{-\gamma}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}} + B_2 + \frac{\left(\frac{b}{\sigma n N^{\mu}}\right)^{q/\nu}}{B_3M^q}\frac{K\sigma^{q/\nu}n^{q/\nu}N^{\mu q/\nu}}{b^{q/\nu}}\right]. \qquad \text{(IV.84)}
\end{aligned}$$

Algebraic manipulation can be used to show that

$$b^{\frac{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}\frac{pq}{\mu q+\nu p}} b^{\frac{-\gamma}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}} = b^0 = 1, \tag{IV.85}$$

and

$$b^{\frac{q/\nu}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}} b^{\frac{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}\frac{\mu q^2}{\mu q\nu+\nu^2 p}} b^{-q/\nu} b^{\frac{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}}{\frac{\mu\gamma}{p}+\frac{\nu\gamma}{q}+1}\frac{pq}{\mu q+\nu p}} = b^0 = 1. \tag{IV.86}$$

Then, since $\sigma n_b N_b^\mu M_b^\nu / b \to 1$, the sum of terms in brackets in (IV.84), with n, N, and M replaced by n_b, N_b, and M_b, respectively, tends to a constant as $b \to \infty$. The conclusion for the second part of the theorem then follows from (IV.84) after taking logarithms, dividing by $\log b$, and taking limits. □

Based on Theorem IV.9, the rate of convergence of the total error bound for a sublinearly convergent optimization algorithm is worse than that which is achievable by finite, superlinear, and linear algorithms (see Theorems IV.4, IV.6, and IV.8), even when the optimal discretization policy given in the second part of the theorem is used. We also note that if γ tends to infinity, then the rate for the sublinear case, when the optimal discretization policy is used, tends to that of the finite, superlinear, and linear cases as it should.

D. CONCLUSIONS

In this chapter, we consider the rate of convergence of a bound on the error between the objective function evaluated at iterates generated from the discretized problems and the optimal value of the original problem as a computational budget b tends to infinity. We see that in the case of superlinear and linear optimization algorithms, the best possible rate of convergence is $b^{-1/(\mu/p+\nu/q)}$, where μ and ν are positive parameters related to the computational work per iteration in the optimization algorithms, and p and q are positive parameters related to the error in the numerical methods used to construct the finite-dimensional problems. We identify specific optimal discretization policies for both the superlinear and linear cases that achieve this best possible rate. It is impossible to improve upon this rate due to

the fact that there is always some level of discretization error. If other discretization policies are utilized, it is possible that they will result in substantially slower rates of convergence. For sublinear optimization algorithms, with optimization error bounded by C/n^γ, $C \geq 0$, $\gamma > 0$, after n iterations, the best possible rate of convergence is $b^{-1/(1/\gamma+\mu/p+\nu/q)}$. This rate is slower than what can be achieved with either a superlinear or linear optimization algorithm. The optimal discretization policy specified in Section IV.C.4 achieves the rate $b^{-1/(1/\gamma+\mu/p+\nu/q)}$. Table 2 gives a summary of the results for the different optimization algorithms we consider in this chapter.

Table 2. Comparison for optimization algorithms.

Optimization Algorithm	Asymptotic rate of decay of error bound
Finite	$b^{\frac{-1}{\mu/p+\nu/q}}$
Superlinear	$b^{\frac{-1}{\mu/p+\nu/q}}$ (with optimal policy)
Linear	$b^{\frac{-1}{\mu/p+\nu/q}}$ (with optimal policy)
Sublinear	$b^{\frac{-1}{1/\gamma+\mu/p+\nu/q}}$ (with optimal policy)

The results from this chapter provide insight regarding the type of numerical method used to solve the differential equations when implementing a discretization algorithm to solve one of the generalized optimal control problems discussed in Chapter III. If a linear or superlinear optimization algorithm $\mathcal{A}$ is used to solve the finite-dimensional optimal control problems, with Euler's method used to numerically approximate the solution of the differential equations and Simpson's rule used to numerically approximate the spatial integration, then $p = 1$ and $q = 4$ and we have $e(\mathcal{A}, n, N, M) = \Delta^n_{NM}(\mathcal{A}) + \frac{K}{N} + \frac{K}{M^4}$. If we assume that Assumption IV.3 holds for the reasons discussed in Section IV.B with $\mu = 1$ and $\nu = 2$, then the best possible rate of convergence is $b^{-2/3}$.

If a Runge-Kutta algorithm is used instead of Euler's method to numerically solve the differential equations, it is clear from an analysis of Runge-Kutta algorithms (see, for example, Nagle & Saff, 1989 pp. 133-134) that although additional function and gradient evaluations are required, Assumption IV.3 could still hold with $\mu = 1$

and $\nu = 2$. If a second-order Runge-Kutta method is used instead of Euler's method to numerically solve the differential equations, then $p = 2$ and $q = 4$ and we have $e(\mathcal{A}, n, N, M) = \Delta_{NM}^{n}(\mathcal{A}) + \frac{K}{N^2} + \frac{K}{M^4}$. The resulting best possible rate of convergence is then b^{-1}. If a fourth-order Runge-Kutta method is used instead of Euler's method to numerically solve the differential equations, then $p = 4$ and $q = 4$ and we have $e(\mathcal{A}, n, N, M) = \Delta_{NM}^{n}(\mathcal{A}) + \frac{K}{N^4} + \frac{K}{M^4}$. The best possible rate of convergence is then $b^{-4/3}$.

In order to establish an upper bound for the type of numerical method used to solve the differential equations, we let $p \to \infty$. Then we have $e(\mathcal{A}, n, N, M) = \Delta_{NM}^{n}(\mathcal{A}) + \frac{K}{M^4}$. If we assume that Assumption IV.3 still holds with $\mu = 1$ and $\nu = 2$, then the best possible rate of convergence is b^{-2}. The analysis of this chapter therefore indicates that higher-order methods such as Runge-Kutta or pseudo-spectral may lead to better overall performance. However, this potential improvement in performance comes at the cost of additional complications in both the theory and implementation.

We can also establish an upper bound for the type of numerical integration method used to evaluate the spatial integration if we let $q \to \infty$. Then we have $e(\mathcal{A}, n, N, M) = \Delta_{NM}^{n}(\mathcal{A}) + \frac{K}{N^p}$. If we assume that Assumption IV.3 still holds with $\mu = 1$ and $\nu = 2$, then the best possible rate of convergence is b^{-p}. The value of p depends on the choice of numerical method used to solve the differential equations, with $p = 1$ for Euler's method, $p = 2$ for a second-order Runge-Kutta method, and $p = 4$ for a fourth-order Runge-Kutta method. Table 3 gives a summary of the results for the different numerical methods used to solve the differential equations and evaluate the spatial integration.

The last row in Table 3 gives the asymptotic rate of decay of the error bound assuming "Ideal" methods are used to solve the differential equations as well as evaluate the spatial integration. The term "Ideal" method means we let $p \to \infty$ and $q \to \infty$. Then we have $e(\mathcal{A}, n, N, M) = \Delta_{NM}^{n}(\mathcal{A})$. There is no benefit associated with

increasing N or M, so $b = n$ and the resulting asymptotic rates are c^{γ^b} and $\bar{c}^b$ for superlinear and linear optimization algorithms, respectively.

Table 3. Comparison for numerical methods used to solve differential equations and evaluate the spatial integration. The optimization algorithm can be finitely, superlinearly, or linearly convergent. The last row in the table gives the asymptotic rate of decay of the error bound assuming "Ideal" methods are used to solve the differential equations as well as evaluate the spatial integration. The rates given are for a superlinear optimization algorithm with order $\gamma \in (1, \infty)$ and $c \in (0, 1)$, and a linear optimization algorithm with rate constant $\bar{c} \in (0, 1)$.

Numerical Method for Differential Equations	Numerical Method for Spatial Integration	Asymptotic rate of decay of error bound
Euler	Simpson	$b^{\frac{-2}{3}}$
2^{nd} Order Runge-Kutta	Simpson	b^{-1}
4^{th} Order Runge-Kutta	Simpson	$b^{\frac{-4}{3}}$
"Ideal" $p \to \infty$	Simpson	b^{-2}
Euler	"Ideal" $q \to \infty$	b^{-1}
2^{nd} Order Runge-Kutta	"Ideal" $q \to \infty$	b^{-2}
4^{th} Order Runge-Kutta	"Ideal" $q \to \infty$	b^{-4}
"Ideal" $p \to \infty$	"Ideal" $q \to \infty$	c^{γ^b} or $\bar{c}^b$

We leave verification of the theoretical rate of convergence results developed in this chapter via numerical examples for future work. We note that this verification will be challenging due to the fact that it will be difficult to determine what the optimal value of the objective function should be for the problems under consideration. The theoretical results of this chapter are still useful, however, as they provide insight regarding the choice of numerical methods for approximately solving the differential equations and approximating the spatial integrals as well as the optimization algorithms that can be utilized to implement a discretization algorithm that solves a generalized optimal control problem.

V. ALGORITHMS AND NUMERICAL RESULTS

This chapter bridges the gap between the theory developed in Chapter III and implementable algorithms that can be used to solve the problems (GTP), (GTP^c), (GTP^e), $(GTP^{c,e})$, (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$. Again, for the sake of brevity we omit explicit treatment of the problems (GTP^e) and $(GTP^{c,e})$ in this chapter. The results developed in this chapter for (ITP^e) and $(ITP^{c,e})$ can again be trivially extended to include (GTP^e) and $(GTP^{c,e})$, respectively, with α^l replaced by α and $\phi^l(\cdot)$ replaced by $\phi(\cdot)$. In Section V.A, we discuss problems defined on a real Euclidean space of coefficients needed to construct implementable algorithms, followed by a statement of the algorithms and their proofs of convergence, where appropriate. Then, in Section V.B.1, we use one of these algorithms, with a fixed discretization scheme, to solve an instance of (GTP^c). This solution is then used to derive operational insights on how to better defend a HVU against an attack from a collection of small boat adversaries. We also give examples of numerical solutions to instances of $(ITP^{c,e})$ and $(ITP^{c,p})$ using fixed discretization schemes. In an effort to develop faster solution methods, in Section V.B.2 we use an algorithm with an adaptive precision-adjustment scheme to solve (GTP^c) and compare the results acheived to those obtained using different fixed discretization schemes. Finally, in Section V.B.3, we compare three heuristic methods designed for use onboard unmanned systems to provide solutions to (GTP^c) in real time.

A. IMPLEMENTABLE ALGORITHMS

The approximating problems (GTP_{NM}), (GTP^c_{NM}), (ITP^e_{NM}), $(ITP^{c,e}_{NM})$, (ITP^p_{NM}), and $(ITP^{c,p}_{NM})$ in Chapter III were all defined using the function space H_N. We would like to use existing nonlinear programming algorithms to solve these approximating generalized optimal control problems. The issue with using these existing

algorithms is that they are defined only on real Euclidean spaces. For this reason, we need to define equivalent problem formulations in a corresponding real Euclidean space of coefficients that we can use to calculate numerical solutions. Equivalence is based on correspondence between solutions as in sections 4.3 and 5.6 of Polak (1997). We adopt the notation used in sections 4.3 and 5.6 of Polak (1997) to define these problems, and mirror the development found on pages 722-723 of Polak (1997) in this section.

If we let e_j denote the jth unit vector in $\mathbb{R}^m$, then the functions $e_j\pi_{N,\kappa}(\cdot)$, $j = 1, ..., m, \kappa = 0, ..., N-1$, with $\pi_{N,\kappa}$ defined as in (III.54), form an orthonormal basis for the subspace L_N defined in (III.55). This means that H_N is in one-to-one correspondence with the real Euclidean space

$$\bar{H}_N \triangleq \mathbb{R}^n \times \mathbb{R}^{mN}. \tag{V.1}$$

The elements of $\bar{H}_N$ are $\bar{\eta} = (\xi, \bar{u})$, with $\xi \in \mathbb{R}^n$ and $\bar{u} = (\bar{u}_0, \bar{u}_1, ..., \bar{u}_{N-1}) \in \mathbb{R}^m \times \mathbb{R}^m \times ... \times \mathbb{R}^m$. Because H_N is in one-to-one correspondence with $\bar{H}_N$, any $\eta = (\xi, u) \in H_N$, with $u(\cdot) = \sum_{\kappa=0}^{N-1} \bar{u}_\kappa \pi_{N,\kappa}(\cdot)$, corresponds to a unique $\bar{\eta} = (\xi, \bar{u}) \in \bar{H}_N$, with $\bar{u}$ split into N, m-dimensional blocks $\bar{u} = (\bar{u}_0, \bar{u}_1, ..., \bar{u}_{N-1})$. Then for any $N \in \mathcal{N}$, we define the linear, invertible maps $W_N : H_N \to \bar{H}_N$ by

$$W_N\left(\xi, \sum_{\kappa=0}^{N-1} \bar{u}_\kappa \pi_{N,\kappa}(\cdot)\right) \triangleq (\xi, (\bar{u}_0, \bar{u}_1, ..., \bar{u}_{N-1}))\,. \tag{V.2}$$

For any $\eta, \eta' \in H_N$ and $\bar{\eta} = W_N(\eta)$, $\bar{\eta}' = W_N(\eta')$,

$$\begin{aligned}
\langle \eta, \eta' \rangle_{H_2} &= \langle \xi, \xi' \rangle + \int_0^1 \langle u(t), u'(t) \rangle\, dt \\
&= \langle \xi, \xi' \rangle + \frac{1}{N} \sum_{\kappa=0}^{N-1} \left\langle \sqrt{N}\bar{u}_\kappa, \sqrt{N}\bar{u}'_\kappa \right\rangle \\
&= \langle \xi, \xi' \rangle + \sum_{\kappa=0}^{N-1} \langle \bar{u}_\kappa, \bar{u}'_\kappa \rangle \triangleq \langle \bar{\eta}, \bar{\eta}' \rangle_{\bar{H}_N}\,,
\end{aligned} \tag{V.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle_{\bar{H}_N}$ denotes the Euclidean inner product on $\bar{H}_N$. Which means for any $\eta \in H_N$ and $\bar{\eta} = W_N(\eta)$,

$$\|\eta\|^2_{H_2} = \langle \eta, \eta \rangle_{H_2} = \|\bar{\eta}\|^2_{\bar{H}_N} = \langle \bar{\eta}, \bar{\eta} \rangle_{\bar{H}_N}\,. \tag{V.4}$$

This shows that $W_N(\cdot)$ is a norm-preserving, or isometric, map whose adjoint is $W_N^{-1}(\cdot)$, which means given any $\eta \in H_N$ and $\bar{\eta} \in \bar{H}_N$,

$$\langle \bar{\eta}, W_N(\eta) \rangle_{\bar{H}_N} = \left\langle W_N^{-1}\left(\bar{\eta}\right), \eta \right\rangle_{H_N}. \tag{V.5}$$

Given an $\bar{\eta} = (\xi, \bar{u}) \in \bar{H}_N$, if we let $\eta = W_N^{-1}\left(\bar{\eta}\right)$, then (III.60) defines the sequence of vectors $\bar{x}_N^k \triangleq \left\{\bar{x}_N^{\bar{\eta},k}(j)\right\}_{j=0}^N$ in $\mathbb{R}^n$, for all $k = 1, 2, ..., K$, by the recursion

$$\begin{aligned} \bar{x}_N^{\bar{\eta},k}\left((j+1)/N\right) - \bar{x}_N^{\bar{\eta},k}\left(j/N\right) &= \frac{1}{N} h^k \left(\bar{x}_N^{\bar{\eta},k}(j/N), \sqrt{N}\bar{u}^k(j/N)\right), \\ j &\in \left\{0, 1, ..., N-1\right\}, \bar{x}_N^{\bar{\eta}}(0) = \xi. \end{aligned} \tag{V.6}$$

Similarly, (III.61) and (III.181) define the sequence of vectors $\bar{z}_N \triangleq \left\{\bar{z}_N^{\bar{\eta}}(j;\alpha)\right\}_{j=0}^N \in \mathbb{R}$ and $\bar{z}_N^l \triangleq \left\{\bar{z}_N^{\bar{\eta},l}(j;\alpha^l)\right\}_{j=0}^N \in \mathbb{R}$ by the recursions

$$\begin{aligned} \bar{z}_N^{\bar{\eta}}\left((j+1)/N;\alpha\right) - \bar{z}_N^{\bar{\eta}}\left(j/N;\alpha\right) &= \frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L} r^{k,l}\left(\bar{x}_N^{\bar{\eta},k}(j/N), y^l\left(j/N;\alpha\right)\right), \\ j &\in \left\{0, 1, ..., N-1\right\}, \bar{z}_N^{\bar{\eta}}(0;\alpha) = 0, \end{aligned} \tag{V.7}$$

and

$$\begin{aligned} \bar{z}_N^{\bar{\eta},l}\left((j+1)/N;\alpha^l\right) - \bar{z}_N^{\bar{\eta},l}\left(j/N;\alpha^l\right) &= \frac{1}{N}\sum_{k=1}^{K} r^{k,l}\left(\bar{x}_N^{\bar{\eta},k}(j/N), y^l\left(j/N;\alpha^l\right)\right), \\ j &\in \left\{0, 1, ..., N-1\right\}, \bar{z}_N^{\bar{\eta},l}(0;\alpha^l) = 0, \end{aligned} \tag{V.8}$$

respectively.

Using the discretized "information state" given by the recursion (V.7), we define the approximate objective functions $\bar{f}_{NM} : \bar{H}_N \to \mathbb{R}$ for any $\bar{\eta} \in \bar{H}_N$ and $N \in \mathcal{N}$ by

$$\bar{f}_{NM}(\bar{\eta}) \triangleq I_M\left(\exp\left[-\bar{z}_N^{\bar{\eta}}(1;\cdot)\right]\phi(\cdot)\right), \tag{V.9}$$

and the corresponding discrete generalized optimal control problems by

$$(\overline{GTP}_{NM}) \quad \min_{\bar{\eta}\in\bar{H}_N} \quad \bar{f}_{NM}(\bar{\eta}), \tag{V.10}$$

and

$$(\overline{GTP}^c_{NM}) \quad \min_{\bar{\eta} \in \bar{\mathbf{H}}_{c,N}} \quad \bar{f}_{NM}(\bar{\eta}), \tag{V.11}$$

where

$$\bar{\mathbf{H}}_{c,N} \triangleq \mathbb{R}^n \times \left\{ \bar{u} = (\bar{u}_0, \bar{u}_1, ..., \bar{u}_{N-1}) \in \mathbb{R}^{mN} | \bar{u}_j \in U, j \in \{0, 1, ..., N-1\} \right\}, \tag{V.12}$$

with U as in (III.9).

Similarly, using the discretized "information state" given by the recursion (V.8), we define the approximate objective functions $\overline{\psi}^e_{NM} : \bar{H}_N \to \mathbb{R}$ and $\overline{\psi}^p_{NM} : \bar{H}_N \to \mathbb{R}$ for any $\bar{\eta} \in \bar{H}_N$ and $N \in \mathcal{N}$ by

$$\overline{\psi}^e_{NM}(\bar{\eta}) \triangleq \sum_{l=1}^{L} I_M \left(\exp \left[-\bar{z}^{\bar{\eta},l}_N(1; \cdot) \right] \phi^l(\cdot) \right), \tag{V.13}$$

and

$$\overline{\psi}^p_{NM}(\bar{\eta}) \triangleq \prod_{l=1}^{L} I_M \left(\exp \left[-\bar{z}^{\bar{\eta},l}_N(1; \cdot) \right] \phi^l(\cdot) \right). \tag{V.14}$$

We define the corresponding discrete generalized optimal control problems by

$$(\overline{ITP}^e_{NM}) \quad \min_{\bar{\eta} \in \bar{H}_N} \quad \overline{\psi}^e_{NM}(\bar{\eta}), \tag{V.15}$$

$$(\overline{ITP}^{c,e}_{NM}) \quad \min_{\bar{\eta} \in \bar{\mathbf{H}}_{c,N}} \quad \overline{\psi}^e_{NM}(\bar{\eta}), \tag{V.16}$$

$$(\overline{ITP}^p_{NM}) \quad \min_{\bar{\eta} \in \bar{H}_N} \quad \overline{\psi}^p_{NM}(\bar{\eta}), \tag{V.17}$$

and

$$(\overline{ITP}^{c,p}_{NM}) \quad \min_{\bar{\eta} \in \bar{\mathbf{H}}_{c,N}} \quad \overline{\psi}^p_{NM}(\bar{\eta}). \tag{V.18}$$

Based on Exercise 4.3.7 in Polak (1997), the problems $(\overline{GTP}_{NM})$, $(\overline{GTP}^c_{NM})$, $(\overline{ITP}^e_{NM})$, $(\overline{ITP}^{c,e}_{NM})$, $(\overline{ITP}^p_{NM})$, and $(\overline{ITP}^{c,p}_{NM})$ are equivalent to the problems (GTP_{NM}), (GTP^c_{NM}), (ITP^e_{NM}), $(ITP^{c,e}_{NM})$, (ITP^p_{NM}), and $(ITP^{c,p}_{NM})$, respectively,

in the sense that they are related by a nonsingular linear transformation, and that at corresponding points the values of their corresponding optimality functions are the same.

We now state generic algorithm models based on Master Algorithm Model 3.3.12 from Polak (1997) that can be used to solve the problems (GTP), (GTP^c), (ITP^e), $(ITP^{c,e})$, (ITP^p), and $(ITP^{c,p})$. Our algorithms define the "outer" iterations, while the "inner" iterations are defined by user supplied iteration maps $\mathcal{A}((P), \bar{\eta})$, where $\mathcal{A}$ denotes one iteration of a nonlinear programming solver applied to problem (P) starting from $\bar{\eta}$. We begin with the unconstrained problems and state the following algorithm, which is based on a fixed discretization scheme.

Algorithm V.1. Approximately solves (GTP).

Data: $N_0 \in \mathcal{N}, M_0 \in \mathbb{N}_3 \times \mathbb{N}_3$[5]$, \eta_0 \in \mathbf{H}^0_{N_0}$.

Step 0. Set $N = N_0$, $M = M_0$, and $\bar{\eta}_0 = W_N(\eta_0)$.

Step 1. Generate $\{\bar{\eta}_i\}_{i=0}^{\infty}$ using $\bar{\eta}_{i+1} \in \mathcal{A}((\overline{GTP}_{NM}), \bar{\eta}_i)$. □

Algorithm V.1 also approximately solves (ITP^e) and (ITP^p), with $(\overline{GTP}_{NM})$ replaced by $(\overline{ITP}^e_{NM})$ and $(\overline{ITP}^p_{NM})$, respectively. For the constrained problems, we have the following algorithm.

Algorithm V.2. Approximately solves (GTP^c).

Data: $N_0 \in \mathcal{N}, M_0 \in \mathbb{N}_3 \times \mathbb{N}_3, \eta_0 \in \mathbf{H}_{c,N_0}$.

Step 0. Set $N = N_0$, $M = M_0$, and $\bar{\eta}_0 = W_N(\eta_0)$.

Step 1. Generate $\{\bar{\eta}_i\}_{i=0}^{\infty}$ using $\bar{\eta}_{i+1} \in \mathcal{A}((\overline{GTP}^c_{NM}), \bar{\eta}_i)$. □

Algorithm V.2 also approximately solves $(ITP^{c,e})$ and $(ITP^{c,p})$, with $(\overline{GTP}^c_{NM})$ replaced by $(\overline{ITP}^{c,e}_{NM})$ and $(\overline{ITP}^{c,p}_{NM})$, respectively.

In order to improve the run-time performance of Algorithms V.1 and V.2, we also develop algorithms based on an adaptive precision-adjustment scheme. Algorithms based on adaptive precision-adjustment keep the discretization parameters, N and M, small when the algorithm starts and is "far" from the optimal solution.

[5]Recall that $\mathbb{N}_3 \triangleq \{m \in 2\mathbb{N} + 1 | m \geq 3\}$, as defined in Proposition III.26.

We use the optimality function to determine how "far" the current iterate is from the optimal solution. The values of N and M are then increased according to successor functions as the algorithm gets closer to the optimal solution. One advantage these algorithms have over fixed precision algorithms is that the computational work remains relatively low until the algorithm gets close to the optimal solution. Another advantage is that solutions obtained using low levels of discretization are used to "warm start" the algorithm at higher discretization levels. Algorithms based on adaptive precision-adjustment fit nicely within the framework of consistent approximations because we know that as the values of N and M tend to infinity, the solution of the approximating problem converges to the solution of the original problem. These algorithms use successor function $\kappa : \mathcal{N} \to \mathcal{N}$ defined by

$$\kappa(N) \in \{N' \in \mathcal{N} | N' > N\} . \tag{V.19}$$

It is clear from (V.19) that $\kappa(N)$ is an integer larger than $N \in \mathcal{N}$. We begin with the unconstrained problems and state the following algorithm, which is based on an adaptive precision-adjustment scheme.

Algorithm V.3. Solves (GTP).

Data: $N_0 \in \mathcal{N}, M(N) : \mathcal{N} \to \mathbb{N}_3 \times \mathbb{N}_3, \eta_0 \in \mathbf{H}^0_{N_0}$.

Parameter: $\beta_1, \beta_2, \beta_3 > 0$.

Step 0. Set $i = 0, N = N_0$, $M = M(N_0)$, and $\bar{\eta}_i = W_N(\eta_i)$.

Step 1. Compute an $\bar{\eta}_{i+1} \in \mathcal{A}((\overline{GTP}_{NM(N)}), \bar{\eta}_i)$.

Step 2. If $\theta(W_N^{-1}(\bar{\eta}_{i+1})) \geq -(\beta_1/N + \beta_2/(M_1(N))^4 + \beta_3/(M_2(N))^4)$, set $\eta^*_N = W_N^{-1}(\bar{\eta}_{i+1})$, and replace N by $\kappa(N)$.

Step 3. Replace i by $i + 1$, and go to Step 1. □

Algorithm V.3 also solves (ITP^e) and (ITP^p), with $(\overline{GTP}_{NM(N)})$ replaced by $(\overline{ITP}^e_{NM(N)})$ and $(\overline{ITP}^p_{NM(N)})$, and $\theta(\cdot)$ replaced by $\theta^e(\cdot)$ and $\theta^p(\cdot)$, respectively. Then the following theorem is a direct consequence of Theorem III.29.

Theorem V.1. *If the sequences $\{\bar{\eta}_i\}_{i=0}^{\infty}$ and $\{\eta^*_N\}$ are constructed by Algorithm V.3, then*

(i) *if the sequence* $\{\eta_N^*\}$ *is finite, then the sequence* $\{\bar{\eta}_i\}_{i=0}^{\infty}$ *has no accumulation points; and*

(ii) *if the sequence* $\{\eta_N^*\}$ *is infinite, then every accumulation point* $\eta^* \in \mathbf{H}^0$ *of* $\{\eta_N^*\}_{i=0}^{\infty}$*, satisfies* $\theta(\eta^*) = 0$. □

When Algorithm V.3 is used to solve (ITP^e) and (ITP^p), a result similar to Theorem V.1 with $\theta(\cdot)$ replaced by $\theta^e(\cdot)$ and $\theta^p(\cdot)$, respectively, follows as a direct consequence of Theorem III.52.

For the constrained problems, we have the following algorithm.

Algorithm V.4. Solves (GTP^c).

Data: $N_0 \in \mathcal{N}, M(N) : \mathcal{N} \to \mathbb{N}_3 \times \mathbb{N}_3, \eta_0 \in \mathbf{H}_{c,N_0}$.

Parameter: $\beta_1, \beta_2, \beta_3 > 0$.

Step 0. Set $i = 0, N = N_0$, $M = M(N_0)$, and $\bar{\eta}_i = W_N(\eta_i)$.

Step 1. Compute an $\bar{\eta}_{i+1} \in \mathcal{A}((\overline{GTP}^c_{NM(N)}), \bar{\eta}_i)$.

Step 2. If $\theta^c(W_N^{-1}(\bar{\eta}_{i+1})) \geq -(\beta_1/N + \beta_2/(M_1(N))^4 + \beta_3/(M_2(N))^4)$, set $\eta_N^* = W_N^{-1}(\bar{\eta}_{i+1})$, and replace N by $\kappa(N)$.

Step 3. Replace i by $i + 1$, and go to Step 1. □

Algorithm V.4 also solves $(ITP^{c,e})$ and $(ITP^{c,p})$, with $(\overline{GTP}^c_{NM(N)})$ replaced by $(\overline{ITP}^{c,e}_{NM(N)})$ and $(\overline{ITP}^{c,p}_{NM(N)})$, and $\theta^c(\cdot)$ replaced by $\theta^{c,e}(\cdot)$ and $\theta^{c,p}(\cdot)$, respectively. Then the following theorem is a direct consequence of Theorem III.30.

Theorem V.2. *If the sequences* $\{\bar{\eta}_i\}_{i=0}^{\infty}$ *and* $\{\eta_N^*\}$ *are constructed by Algorithm V.4, then*

(i) *if the sequence* $\{\eta_N^*\}$ *is finite, then the sequence* $\{\bar{\eta}_i\}_{i=0}^{\infty}$ *has no accumulation points; and*

(ii) *if the sequence* $\{\eta_N^*\}$ *is infinite, then every accumulation point* $\eta^* \in \mathbf{H}_c$ *of* $\{\eta_N^*\}_{i=0}^{\infty}$*, satisfies* $\theta^c(\eta^*) = 0$. □

When Algorithm V.4 is used to solve $(ITP^{c,e})$ and $(ITP^{c,p})$, a result similar to Theorem V.2 with $\theta^c(\cdot)$ replaced by $\theta^{c,e}(\cdot)$ and $\theta^{c,p}(\cdot)$, respectively, follows as a direct consequence of Theorem III.53.

Theorems V.1 and V.2 are the end result of the theory we develop in Chapter III. Theorems V.1 and V.2 establish the fact that the discretization schemes we develop in Chapter III result in implementable algorithms that are guaranteed to converge to stationary points of the original problems.

B. NUMERICAL RESULTS

1. Fixed Discretization Schemes

In this section we present numerical results based on the situational description given in Chapter II. We start by solving three problem instances given in Table 4 using Algorithm V.2, with parameter values given in Table 5. The column headings K and L in Table 4 denote the number of searchers and targets, respectively. The column heading $\bar{\eta}_0$ in Table 5 represents the initial "guess" for the algorithm. The first three elements of $\bar{\eta}_0$ are the initial headings of the searchers measured from the horizontal axis. The initial control portion of $\bar{\eta}_0$ is $\vec{0}$, which represents the zero vector of length N_0. We note that we do not include the initial position of the searchers in $\bar{\eta}_0$, as they are assumed to be fixed and not part of the decision vector. The initial positions for the searchers are provided in Table 7. The parameter values given in the first row of Table 5 are used with Algorithm V.2 for ProbA and ProbC, and the parameter values given in the second row are used with Algorithm V.2 for ProbB.

Table 4. Fixed discretization problem instances.

Instance	Problem Class	K	L
ProbA	(GTP^c)	3	10
ProbB	$(ITP^{c,e})$	3	2
ProbC	$(ITP^{c,p})$	3	1

Table 5. Algorithm V.2 parameters.

N_0	M_0	$\bar{\eta}_0$
200	(25,25)	$(\pi/4, \pi/2, \pi/2, \vec{0})$
320	(25,25)	$(\pi/4, \pi/2, \pi/2, \vec{0})$

In ProbA, ProbB, and ProbC we consider the case of a HVU operating in a two-dimensional area of interest (AOI), measuring 70 nautical miles (nm) by 70 nm. The HVU follows a straight line trajectory, with constant heading x_3^0 and constant speed v^0 from its initial position $x^0(0)$. The parameter values for the HVU are given in Table 6. While in transit, the HVU is under threat of attack from L targets. The target trajectories are conditioned upon the random vector, α, with realization in $A \subset \mathbb{R}^2$. For our numerical results, the target trajectories are conditioned upon a random starting time between zero and one hour, and a random starting location on either of the vertical sides of the AOI. The numerical solutions we find for ProbA, ProbB, and ProbC are all based on a receding horizon search, where we are planning for the next hour.

To generate numerical solutions for ProbA, ProbB, and ProbC, we implement and run Algorithm V.2 in MATLAB 7.11.0 (version 2010b) (see MathWorks, 2011) on a 3.46 GHz PC with two quad-core processors, using Windows 7 Pro 64-bit, with 24 GB of RAM. We use the SQP algorithm in TOMLAB SNOPT solver (see Gill et al., 2007), with the default major and minor optimality tolerance as the stopping criteria.

In ProbA, the HVU is under threat of attack from $L = 10$ targets. We assume that the distributions for the targets' starting time and starting location are uniform and independent. The range of the uniform distribution for starting location is $[0, 140]$, as we combine both sides of the AOI into a single segment. The range of the uniform distribution for the starting time is $[0, 1]$. The targets follow deterministic trajectories, $\left\{y^l(t;\alpha) : 0 \leq t \leq 1\right\}$, given α, where $y^l(t;\alpha) = (y_1^l(t;\alpha), y_2^l(t;\alpha))^T \in \mathbb{R}^2$ is the position of the l^{th} target at time t.

The target trajectories are generated by solving the optimal control problem described in Section II.C with an additional constraint for each target. The additional constraint requires the targets to hit the HVU at a 90 degree angle of incidence at the final time, t_f. The parameter values used to formulate this optimal control problem

are given in Table 6. The starting position for the l^{th} target, $y_0^l(\alpha)$, is on one of the vertical sides of the AOI, and is determined based on the level of spatial discretization. We assume that the separation in starting position between members of the swarm is 1.5 nm. The target trajectories are truncated because they are generated by solving

Table 6. Target and HVU parameter values. The target parameter values are the same for all targets $l = 1, 2, ..., L$.

v_0^l	v_f^l	v_{min}^l	v_{max}^l	$\bar{u}^{l,tar}$	$x^0(0)$	x_3^0	v^0
15 kts	19 kts	1 kt	35 kts	250 $\frac{\text{rad}}{\text{hour}}$	(35,0) (nm,nm)	$\pi/2$	25 kts

a time-optimal control problem, and our search problem uses a receding horizon approach based on the planning horizon $[0, 1]$. An example of this truncation is illustrated in Figure 3, where the trajectory of the HVU is indicated in blue, the first component of the vector M gives the number of starting locations for the swarm, and the second component of the vector M gives the number of starting times for the swarm. The graph on the top left of Figure 3 shows the trajectories for a single swarm consisting of $L = 10$ targets. The middle graph in the top row of Figure 3 indicates the different trajectories that the swarm would follow if it started from five specific locations. The trajectories for the different starting locations are shown in alternating colors, red and then cyan. The graph on the top right of Figure 3 shows the different trajectories that the swarm would follow if it started from five specific locations and two specific starting times. The trajectories for the different starting locations are shown in alternating colors, red and then cyan, for the first starting time. The trajectories for the different starting locations are shown in alternating colors, green and then purple, for the second starting time. The graphs on the bottom row of Figure 3 are the same as their counterparts in the top row, except they are truncated to show how far the targets (and HVU) can progress on their desired trajectories during the 1 hour planning horizon.

In an effort to detect the targets, the HVU has an escort consisting of $K = 3$ searchers. The goal of the searchers is to minimize the probability of failing to detect

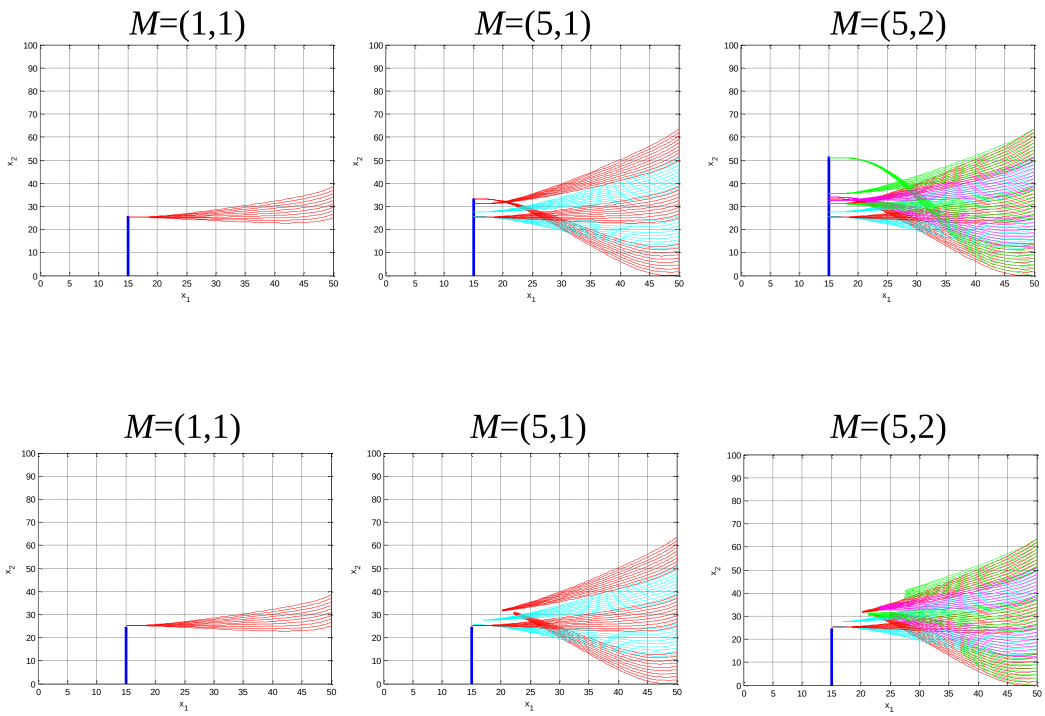

Figure 3. Full and truncated target trajectories for $L = 10$. Top Row Left: $M = (1, 1)$ Full, Top Row Middle: $M = (5, 1)$ Full, Top Row Right: $M = (5, 2)$ Full. Bottom Row Left: $M = (1, 1)$ Truncated, Bottom Row Middle: $M = (5, 1)$ Truncated, Bottom Row Right: $M = (5, 2)$ Truncated.

any of the targets within the planning horizon $[0, 1]$. For any $t \in [0, 1]$, let $x^k(t) = (x_1^k(t), x_2^k(t), x_3^k(t))^T \in \mathbb{R}^3$ be the physical state of the k^{th} searcher at time t, where $x_1^k(t) \in \mathbb{R}$ and $x_2^k(t) \in \mathbb{R}$ are the horizontal and vertical components of the location of the k^{th} searcher, respectively, and $x_3^k(t) \in \mathbb{R}$ is the heading of the k^{th} searcher measured from the horizontal axis. We assume that the searchers are Dubins vehicles traveling at a constant velocity $v^k > 0$. The control input for the k^{th} searcher, $u^k \in \mathbb{R}$, is the rate of change of the heading (also known as the yaw rate) of the k^{th} searcher.

Then, the searcher dynamics described in general terms by (II.22) are given by

$$h^k(x^k(t), u^k(t)) \triangleq \begin{pmatrix} v^k \cos x_3^k(t) \\ v^k \sin x_3^k(t) \\ u^k(t) \end{pmatrix}, k = 1, 2, 3. \tag{V.20}$$

We consider two different search platforms for our numerical results. One is modeled after an SH-60 helicopter, and the other is modeled after a guided missile destroyer (DDG). The searcher parameters are given in Table 7, where $k = 1$ is the helicopter and $k = 2, 3$ are the destroyers. The initial heading of the k^{th} searcher is given by ξ^k, and the initial position of the k^{th} searcher is given by $x^k(0)$. The yaw rate limit for the DDG is based on discussions with CDR David Schiffman, USN, a surface warfare officer in the Operations Research Department at NPS (personal communication, November 2010), and the yaw rate limit for the helicopter searcher is based on the limit for the SH-60 helicopter found in A1-H60BB-NFM-000 (SH-60 NATOPS Flight Manual, 1996).

Table 7. Searcher parameter values.

	$k = 1$	$k = 2$	$k = 3$
ξ^k	$\pi/4$	$\pi/2$	$\pi/2$
v^k	120 kts	25 kts	25 kts
$x^k(0)$	(35,0) (nm,nm)	(25,0) (nm,nm)	(45,0) (nm,nm)
Yaw Rate Limit	1885 $\frac{\text{rad}}{\text{hour}}$	250 $\frac{\text{rad}}{\text{hour}}$	250 $\frac{\text{rad}}{\text{hour}}$

We let $r^{k,l}(x^k(t), y^l(t;\alpha)) \geq 0$ denote the detection rate at time t for the k^{th} searcher at $(x_1^k(t), x_2^k(t))^T$ when the l^{th} target is located at $y^l(t;\alpha)$. For our numerical results, we adopt the Poisson Scan Model (see, e.g., p. 3-1 in Washburn, 2002) and set

$$r^{k,l}((x_1^k(t), x_2^k(t))^T, y^l(t;\alpha)) = \lambda^k \Phi[(F^k - \rho((x_1^k(t), x_2^k(t))^T, y^l(t;\alpha)))/\sigma^k], \tag{V.21}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, λ^k is the scan opportunity rate, F^k is a sensor parameter, σ^k reflects the variability in the received

signal strength, and $\rho^k((x_1^k(t), x_2^k(t))^T, y^l(t;\alpha))$ is used to model the signal loss, which depends on the distance between the target and the searcher; see for example Figure 4.5 on page 93 in Wagner et al. (1999). The detection rate functions for the helicopter searcher, shown in Figure 4, and the DDG searchers, shown in Figure 5, are based on $\rho^k((x_1^k(t), x_2^k(t))^T, y^l(t;\alpha)) = a^k \|(x_1^k(t), x_2^k(t))^T - y^l(t;\alpha)\|^2 + b^k$, with parameter values given in Table 8. The parameter values are chosen to reflect reasonable sensor performance based on discussions with other naval officers in the Operations Research Department at NPS who have served on these platforms (CDR David Schiffman, USN, and CDR Douglas Burton, USN, personal communication, November 2010).

Table 8. Detection rate parameter values.

	$k=1$	$k=2$	$k=3$
λ^k	1.1	1	1
F^k	90	90	90
a^k	0.3	0.3	0.3
b^k	20	60	60
σ^k	100	15	15

The dimension of the decision vector, $\bar{\eta}(\cdot)$, in ProbA is $KN_0+K = 603$, because we determine a control input for each of the searchers at every time step as well as the optimized initial heading for each searcher. Before we give our numerical results, we first show baseline results based on the current concept of operations (CONOPS) in use by the U.S. Navy. This CONOPS has the helicopter escort orbiting the HVU at a distance of 2.5 nm and the DDG escorts keeping station on parallel courses to the HVU at a distance of 10 nm. The CONOPS is based on discussions with helicopter pilots and surface warfare officers in the Operations Research Department at NPS (LCDR Harrison Schramm, USN, LCDR Ron Cappellini, USN, and CDR David Schiffman, USN, personal communication, February 2011). The resulting trajectories are shown in Figure 6. In Figure 6, the target trajectories are plotted in alternating colors of red and cyan to separate the specific starting positions of the swarm. It should be noted that only the set of target trajectories based on a starting time of $t = 0$ and all

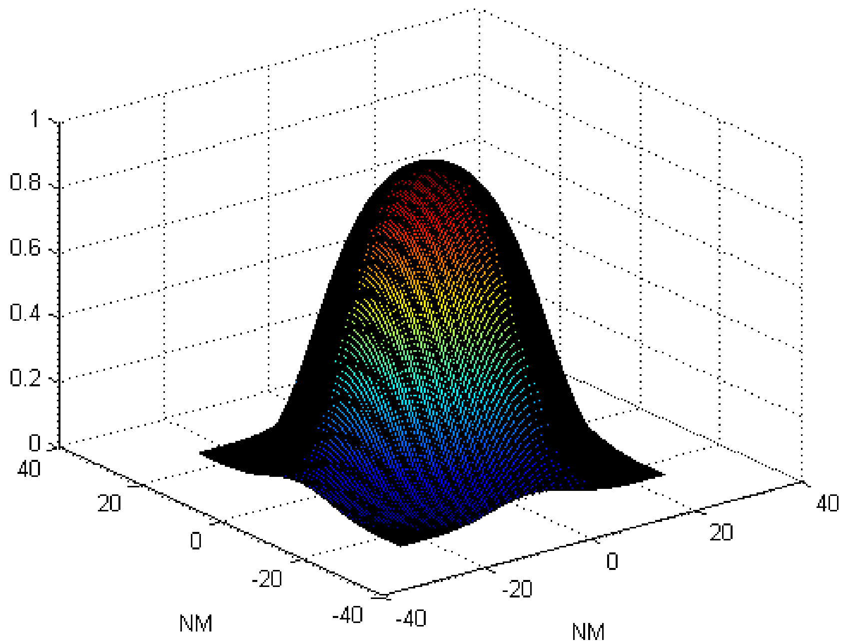

Figure 4. Detection Rate Function for Helo Searcher.

starting locations is plotted in Figure 6. The additional target trajectories based on other starting times between $[0, 1]$ are not shown in the interest of keeping the plot legible. The trajectory for the HVU is shown in blue, the DDG escort trajectories are shown in black, and the helicopter escort trajectory is shown in green. The objective value, which represents the probability that all of the searchers fail to detect any of the targets in $[0, 1]$, for this set of trajectories is 0.9589. It should be noted that as the targets get closer to the HVU the probability that all of the searchers fail to detect any of the targets will be lower, but this improvement might come too late for the HVU to effectively defend itself.

The resulting trajectories from Algorithm V.2 are shown in Figure 7. The colors for the trajectories are the same as those described above for Figure 6. The helicopter trajectory is particularly good at illustrating how the algorithm gets the searchers to areas where they can accumulate as much probability "mass" as possible,

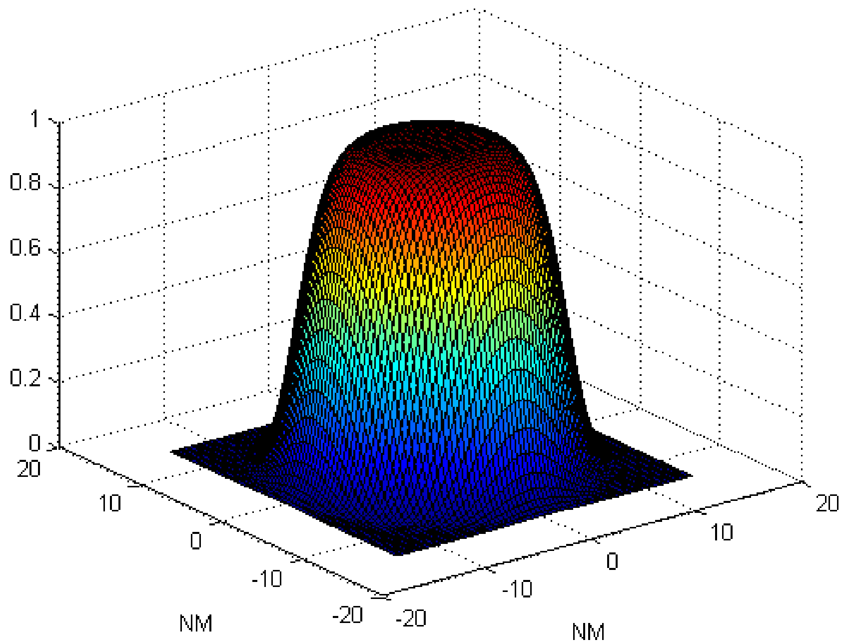

Figure 5. Detection Rate Function for DDG Searcher.

and keeps them there to the extent allowed by the constant speed and turn rate constraints. The objective value is now 0.3954, which is significantly better than the baseline. This indicates there is a benefit to leaving the vicinity of the HVU and seeking out the potential attackers. The time required to solve ProbA was 26.82 hours, including 1.54 hours to generate the target trajectories.

In ProbB the HVU is under threat of attack from $L = 2$ targets. The target $l = 1$ is assumed to leave from the right-hand side of the area of interest and target $l = 2$ is assumed to leave from the left-hand side of the area of interest. The target trajectories are again conditioned upon a random starting time between zero and one hour, and a random starting location along the appropriate vertical side of the AOI. We assume that the target's starting time and location are independent random variables. We assume that the distribution for starting position for both targets is uniform with range $[0, 70]$. We assume that the distribution for starting time for

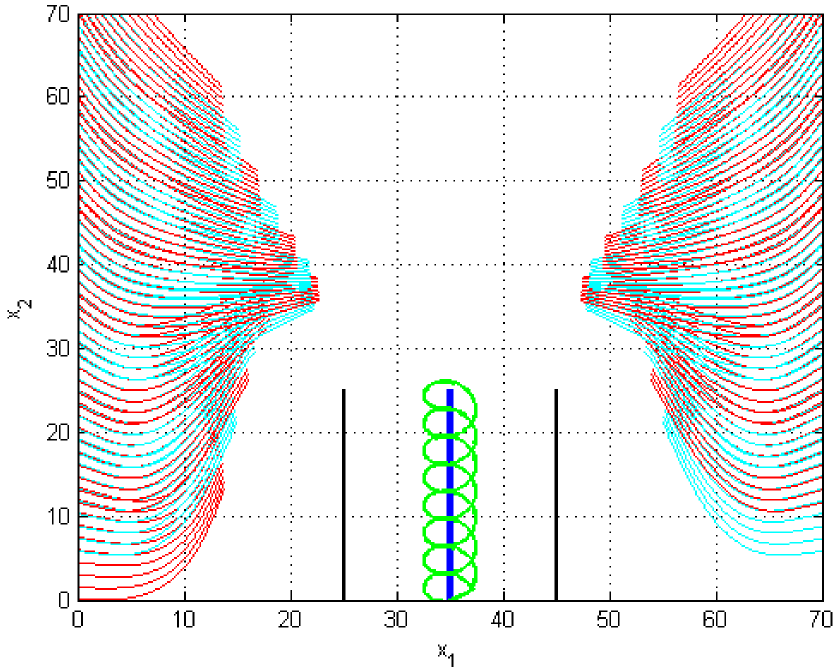

Figure 6. Trajectories based on current CONOPS. HVU trajectory is blue. Helicopter trajectory is green. DDG trajectories are black. Target trajectories alternate red and cyan.

target $l = 1$ is triangular, with the target being twice as likely to leave at $t = 0$ as it is to leave at $t = 1$. We assume that the distribution for starting time for target $l = 2$ is uniform with range $[0, 1]$.

The target trajectories are generated in the same manner as they were for ProbA with the parameter values given in Table 9.

Table 9. Target and HVU parameter values. The target parameter values are the same for targets $l = 1, 2$.

v_0^l	v_f^l	v_{min}^l	v_{max}^l	$\bar{u}^{l,tar}$	$x^0(0)$	x_3^0	v^0
10 kts	12 kts	1 kt	35 kts	250 $\frac{\text{rad}}{\text{hour}}$	(35,0) (nm,nm)	$\pi/2$	25 kts

The HVU again has an escort consisting of $K = 3$ searchers. The goal of the

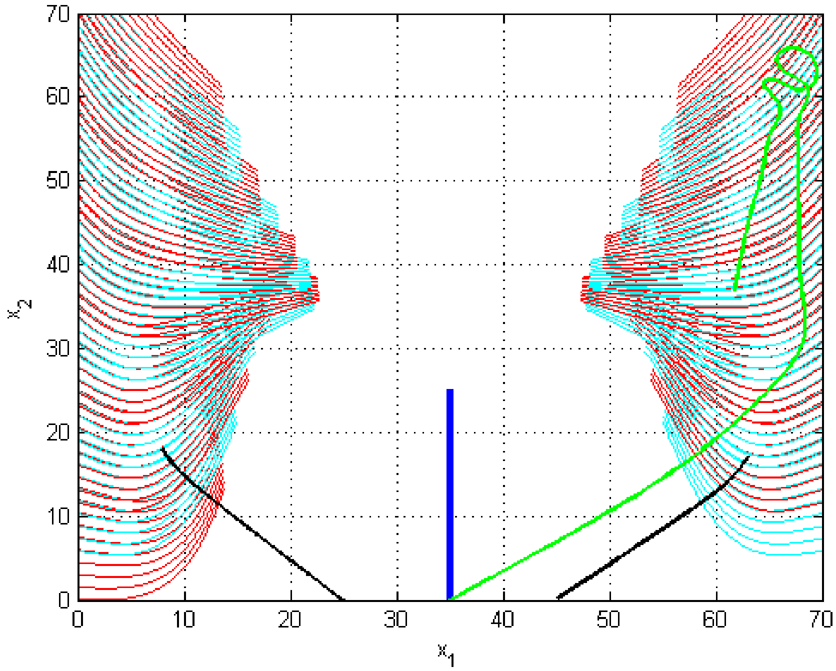

Figure 7. Trajectories based on Algorithm V.2 on ProbA. HVU trajectory is blue. Helicopter trajectory is green. DDG trajectories are black. Target trajectories alternate red and cyan.

searchers is to maximize the expected number of targets detected within the planning horizon $[0, 1]$. The searcher and detection rate parameters are the same as those given in Tables 7 and 8, respectively.

The dimension of the decision vector, $\bar{\eta}(\cdot)$, in ProbB is again given by $KN_0 + K = 963$, which is different value than it was for ProbA due to the different parameter values used in Algorithm V.2. The resulting trajectories from Algorithm V.2 are shown in Figure 8. The color codes for the trajectories are the same as described above for Figure 6, except that target $l = 1$ trajectories are shown in red and target $l = 2$ trajectories are shown in magenta. Again, we note that only the set of target trajectories based on a starting time of $t = 0$ and all starting locations is plotted in Figure 8. The objective value, which represents the expected number of targets

detected in $[0, 1]$, for this set of trajectories is 0.4542. The time required to solve ProbB was 28.71 hours, including 0.33 hours to generate the target trajectories.

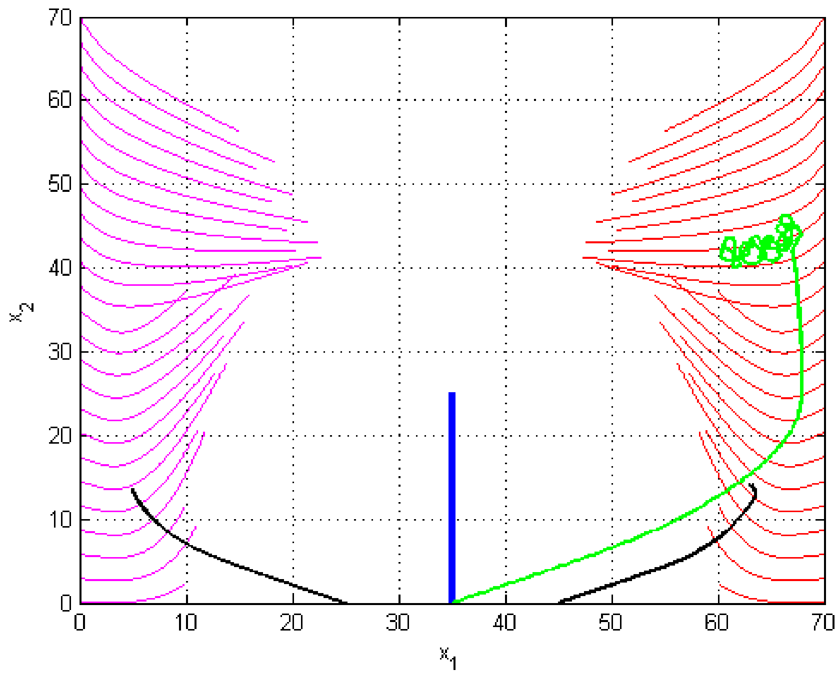

Figure 8. Trajectories based on Algorithm V.2 on ProbB. HVU trajectory is blue. Helicopter trajectory is green. DDG trajectories are black. Target trajectories are magenta and red.

In ProbC the HVU is under threat of attack from $L = 1$ target. The target is assumed to leave from one of the two vertical sides of the area of interest. The target trajectories are conditioned in the same way as they were for ProbA and ProbB. We assume that the distributions for starting time and starting location are independent and uniform. The range of the uniform distribution for starting location is $[0, 140]$, as we again combine both sides of the AOI into a single segment. The range of the uniform distribution for starting time is $[0, 1]$.

The target trajectories are generated in the same manner as they were for

ProbA with the parameter values given in Table 10. We note that we now consider a high-speed target, with a maximum speed of 60 kts.

Table 10. Target and HVU parameter values.

v_0^1	v_f^1	v_{min}^1	v_{max}^1	$\bar{u}^{1,tar}$	$x^0(0)$	x_3^0	v^0
10 kts	12 kts	1 kt	60 kts	250 $\frac{\text{rad}}{\text{hour}}$	(35,0) (nm,nm)	$\pi/2$	25 kts

The HVU again has an escort consisting of $K = 3$ searchers. The goal of the searcher is to minimize the probability of failing to detect any of the targets during the planning horizon $[0, 1]$. The searcher and detection rate parameters are the same as those given in Tables 7 and 8, respectively.

The dimension of the decision vector, $\bar{\eta}(\cdot)$, in ProbC is $KN_0 + K = 603$. The resulting trajectories from Algorithm V.2 are shown in Figure 9. The color codes for the trajectories are the same as described above for Figure 6, except that the target trajectories are shown in red. Again, we note that only the set of target trajectories based on a starting time of $t = 0$ and all starting locations is plotted in Figure 9. The objective value, which represents the probability that all of the searchers fail to detect any of the targets during $[0, 1]$, for this set of trajectories is 0.7979. The time required to solve ProbC was 2.59 hours, including 0.15 hours to generate the target trajectories.

The results in this section show that Algorithm V.2 is tractable for as many as three searchers and ten targets. The results also indicate that it is beneficial for the searchers to leave the vicinity of the HVU and seek out potential threats in order to improve the overall probability of detection or expected number of targets detected. The results of ProbC illustrate the effect of a high-speed target. The helicopter searcher does not spend too much time on one side of the AOI, but rather attempts to cover what it can on one side and then flies over to the other side of the AOI. Finally, we see that obtaining numerical solutions for ProbA, ProbB, and ProbC using a fixed discretization scheme requires a significant amount of computing

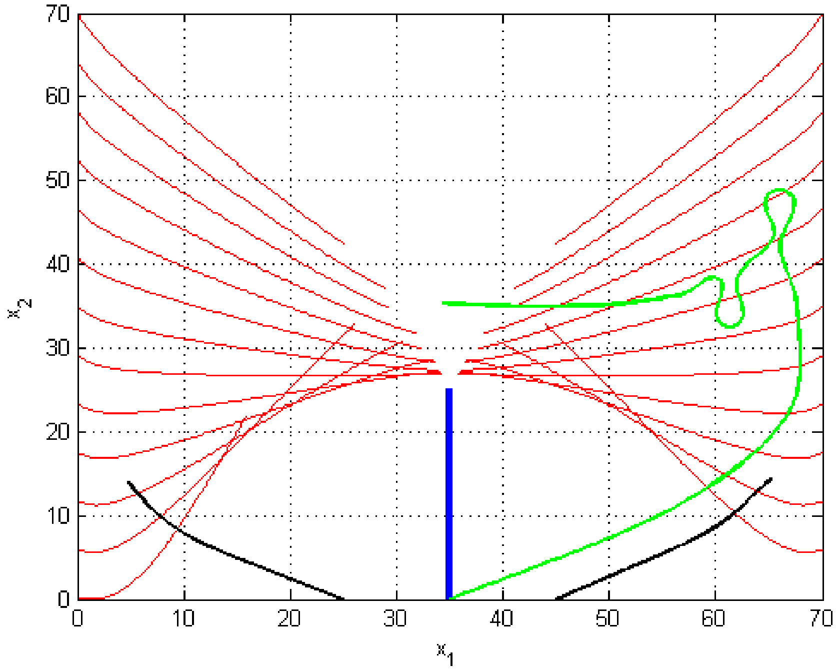

Figure 9. Trajectories based on Algorithm V.2 on ProbC. HVU trajectory is blue. Helicopter trajectory is green. DDG trajectories are black. Target trajectories are red.

time. There is evidence that using an adaptive discretization scheme can produce solutions that are equivalent to those obtained using fixed discretization schemes, but at a lower computational cost; see, for example, He & Polak (1990) and Section 3.3.3 in Polak (1997). We consider an algorithm based on an adaptive discretization scheme in the next section.

2. Adaptive Discretization Schemes

In this section we obtain numerical solutions for ProbA using an adaptive discretization scheme based on Algorithm V.4 and compare them to results obtained for ProbA using the fixed discretization scheme of Algorithm V.2. The parameters β_1, β_2, and β_3 used to define what it means for the optimality function to be "close enough" to zero, the successor function, and the function $M(N)$ used to specify

the level of spatial discretization as a function of the time discretization parameter considered in this section are summarized in Table 11. The column heading Set in Table 11 refers to the set of algorithm parameters used to obtain the solution for ProbA using either Algorithm V.2 or V.4, as indicated in the table. We note that the expressions for $M_i(N)$ given for Algorithm V.4 indicate that the initial spatial discretization parameters are $(5, 5)$ and that they are each increased by adding 2 (for Sets 4 and 6) or 4 (for Sets 5 and 7) to the previous value every time the value of the time discretization parameter is doubled according to $\kappa(N)$.

Table 11. Algorithm parameters used to obtain solutions for ProbA, where $K = 3$, $L = 10$, and $i = 1, 2$. For all Sets, $\bar{\eta}_0 = (\pi/4, \pi/2, \pi/2, \vec{0})$.

Set	Algo.	N_0	M_0	β_1	β_2	β_3	$\kappa(N)$	$M_i(N)$
1	V.2	80	$(13, 13)$	N/A	N/A	N/A	N/A	N/A
2	V.2	120	$(17, 17)$	N/A	N/A	N/A	N/A	N/A
3	V.2	320	$(25, 25)$	N/A	N/A	N/A	N/A	N/A
4	V.4	10	N/A	10^{-9}	5×10^{-10}	5×10^{-10}	$N' = 2N$	$5 + 2\log_2(\frac{N}{10})$
5	V.4	10	N/A	10^{-9}	5×10^{-10}	5×10^{-10}	$N' = 2N$	$5 + 4\log_2(\frac{N}{10})$
6	V.4	10	N/A	10^{-8}	5×10^{-9}	5×10^{-9}	$N' = 2N$	$5 + 2\log_2(\frac{N}{10})$
7	V.4	10	N/A	10^{-8}	5×10^{-9}	5×10^{-9}	$N' = 2N$	$5 + 4\log_2(\frac{N}{10})$

The solutions we obtain are based on the same scenario described in Section V.B.1 for ProbA. The target trajectories are conditioned and generated as described above for ProbA using parameter values given in Table 6. The goal of the $K = 3$ searchers is to minimize the probability that they all fail to detect any of the targets during $[0, 1]$. The searcher and detection rate parameters are the same as those given in Tables 7 and 8, respectively.

The data collected by using Algorithms V.2 and V.4 is shown in Figure 10. The objective function values for all seven algorithm parameter sets were evaluated using a "high precision" discretization level of $N = 2560$ and $M = (91, 91)$ to ensure they were all compared against a common standard. It should be noted that when using the adaptive discretization scheme it is possible for the objective function value

to go up when the algorithm increases the values of the discretization parameters. An example of this behavior is seen for Set 6 after approximately 10000 seconds, where the algorithm increases N from 40 to 80 and M from $(9, 9)$ to $(11, 11)$. Based on Figure 10, it is clear that all of the adaptive precision schemes eventually achieve a solution quality within 1.42% of the best solution obtained using one of the fixed discretization schemes. While the fixed discretization scheme for Set 2 does better than the adaptive discretization schemes after approximately 4000 seconds, it is important to note that all of the adaptive discretization schemes reach an objective value between 0.385 and 0.39 more than a thousand seconds before the fixed discretization scheme for Set 2 does. It is also important to note that all of the adaptive schemes perform better than the fixed schemes based on Sets 1 and 3. This implies that it is possible to do better for a specific problem if you happen to select a "good" fixed discretization scheme at the outset, but doing so without any prior knowledge about the problem is difficult. The adaptive schemes are fairly robust in the sense that they do well regardless of the initial choice of parameter values. The results in Figure 10 indicate that the adaptive precision schemes for Sets 5 and 7 offered the best overall performance compared to the other adaptive discretization schemes.

Figure 11 shows the resulting trajectories from Algorithms V.2 and V.4. The color codes for the trajectories are the same as described above for Figure 6. Again, we note that only the set of target trajectories based on a starting time of $t = 0$ and all starting locations is plotted in Figure 11. It is clear from the plots in Figure 11 that the overall "shape" of the final searcher trajectories is the same regardless of the algorithm and parameters used to obtain the solution, with the caveat that the helicopter searcher trajectory is the mirror image when it goes to the opposite side of the AOI. The times used to plot the trajectories in Figure 11 for the fixed discretization schemes (Sets 1-3) were selected so that enough computational time had elapsed to allow the solution to stabilize. For the adaptive discretization schemes (Sets 4-7), the trajectories were only captured when the algorithm increased the

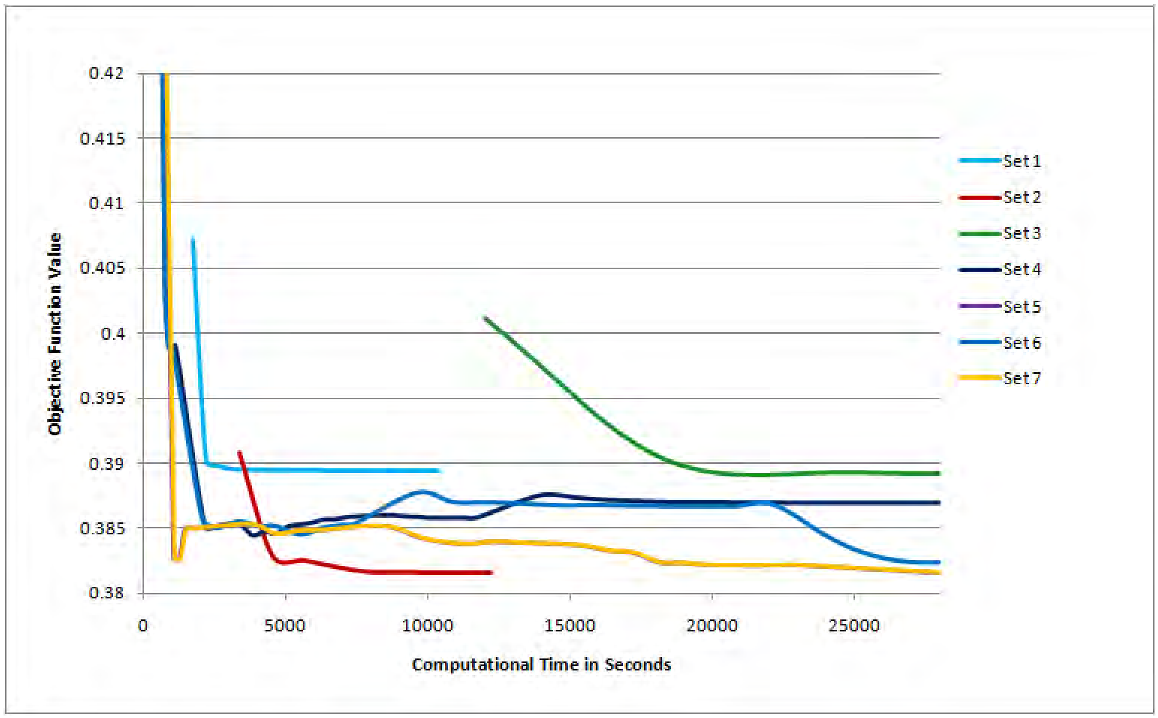

Figure 10. Comparison of fixed and adaptive precision schemes using Algorithms V.2 and V.4, respectively. For Sets 1 and 2 computation was terminated between 10000 and 15000 seconds because the solution had stabilized. Because Set 3 did not begin to stabilize until after 20000 seconds, the horizontal axis was extended.

discretization level. Because of this limitation, the times used to plot the trajectories in Figure 11 for the adaptive discretization schemes were selected based on the largest computational time data that was available.

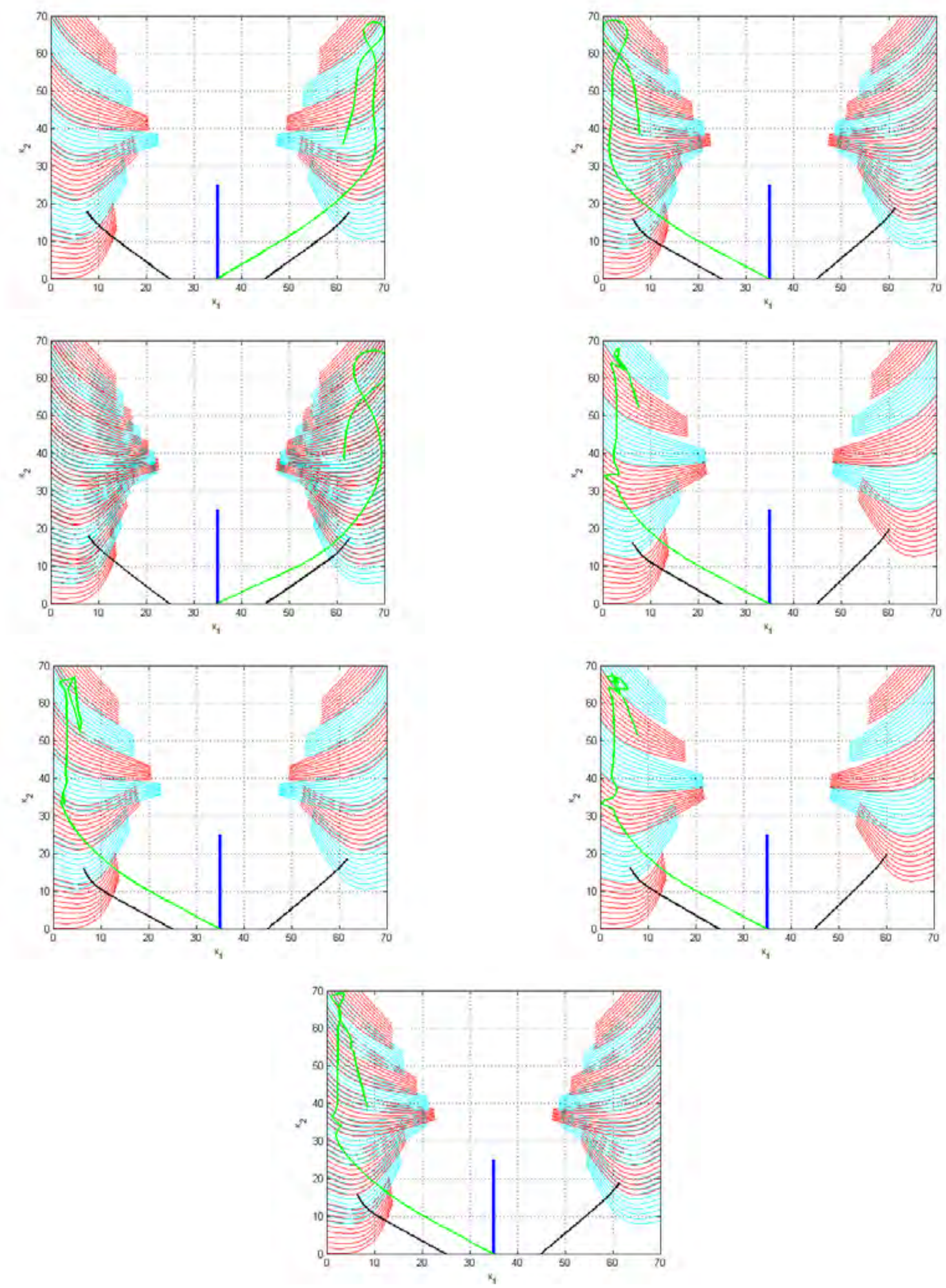

Figure 11. Trajectories for ProbA based on Algorithms V.2 and V.4. Top Row Left: Set 1 after 10339 seconds, $N = 80$, $M = (13, 13)$. Top Row Right: Set 2 after 12201 seconds, $N = 120$, $M = (17, 17)$. Second Row Left: Set 3 after 25147 seconds, $N = 320$, $M = (25, 25)$. Second Row Right: Set 4 after 29852 seconds, $N = 80$, $M = (11, 11)$. Third Row Left: Set 5 after 23152 seconds, $N = 40$, $M = (13, 13)$. Third Row Right: Set 6 after 22167 seconds, $N = 80$, $M = (11, 11)$. Bottom Row Middle: Set 7 after 39998 seconds, $N = 80$, $M = (17, 17)$.

3. Real-Time Methods

Although the adaptive discretization schemes discussed in Section V.B.2 offer improved run-time performance over fixed discretization schemes, they are still not fast enough to be implemented as real-time solution methods for use onboard a UAS or USV. In this section we consider three heuristic approaches that have the potential to be used as real-time solution methods onboard unmanned vehicles. This section begins with a discussion of how we implement the three heuristic methods. We then give a description of how the problem instances to be solved by these three methods were generated. Next we provide numerical results for the three methods. Finally, we state our conclusions based on the numerical results achieved by the three methods.

a. *Heuristic Methods*

We consider three different heuristic methods in this section that can be used to solve generalized optimal control problems. For simplicity we assume that we have only a single searcher, and we therefore omit the superscript notation for the searcher number. It should be noted that the development that follows could be trivially extended to include multiple searchers. The first method we consider uses Algorithm V.2. The second and third methods are based on fitting polynomials to determine optimal searcher trajectories. The first polynomial-based method attempts to optimize directly over the coefficients of the polynomials, while the second polynomial based method uses an indirect method similar to that found in Yakimenko (2000) and Ghabcheloo et al. (2009). The first polynomial-based method solves the following modified version of (GTP^c).

$$(GTP^{c,dir}):$$
$$\min_{a} \left\{ \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} W_{i,j} \exp\left(-\frac{1}{N} \sum_{\gamma=0}^{N-1} \sum_{l=1}^{L} r^l \left(x\left(\frac{\gamma}{N}\right), y^l\left(\frac{\gamma}{N}; \alpha_{i,j}\right)\right)\right) \phi(\alpha_{i,j}) \right\}$$

$$\text{s.t. } x_j\left(\frac{\gamma}{N}\right) = \sum_{i=0}^{d} a_j^i \left(\frac{\gamma}{N}\right)^i, \ \gamma = 0, ..., N-1, \ j = 1, 2 \quad \text{(V.22)}$$

$$\dot{x}_j\left(\frac{\gamma}{N}\right) = \sum_{i=1}^{d} i a_j^i \left(\frac{\gamma}{N}\right)^{i-1}, \ \gamma = 0, ..., N-1, \ j = 1, 2 \quad \text{(V.23)}$$

$$v - 0.05v \leq \left\| \left(\dot{x}_1\left(\frac{\gamma}{N}\right), \dot{x}_2\left(\frac{\gamma}{N}\right)\right) \right\| \leq v + 0.05v, \ \gamma = 0, ..., N-1 \quad \text{(V.24)}$$

$$a = \left[a_1^1, a_1^2, ..., a_1^d; a_2^1, a_2^2, ..., a_2^d\right] \quad \text{(V.25)}$$

$$a_1^0 = x_1(0) \quad \text{(V.26)}$$

$$a_2^0 = x_2(0), \quad \text{(V.27)}$$

where $W_{i,j}$ are the weights for Simpson's rule, $x\left(\frac{\gamma}{N}\right) = \left(x_1\left(\frac{\gamma}{N}\right), x_2\left(\frac{\gamma}{N}\right)\right)^T$, $\alpha_{i,j}$ are the discretization points at which the integrand is evaluated, d is the order of the polynomials, v is the searcher's speed, and $x_1(0)$ and $x_2(0)$ are the initial position of the searcher. We note that (V.24) constrains the searcher's speed to be within 5% of v.

We then use the following algorithm to solve $(GTP^{c,dir})$.

Algorithm V.5. Approximately solves $(GTP^{c,dir})$.

Data: $N_0 \in \mathcal{N}, M_0 \in \mathbb{N}_3 \times \mathbb{N}_3, a_0 \in \mathbb{R}^2 \times \mathbb{R}^2 \times ... \times \mathbb{R}^2$.

Parameter: $d \in \mathbb{N}, d \geq 1$.

Step 0. Set $N = N_0$, $M = M_0$, and $a = a_0$.

Step 1. Generate $\{a_i\}_{i=0}^{\infty}$ using $a_{i+1} \in \mathcal{A}((GTP^{c,dir}), a_i)$. □

The second polynomial based method uses an indirect method to determine the coefficients of the polynomials. We use the term indirect because instead of directly optimizing over the polynomial coefficients, we instead optimize over terminal conditions which can be used to obtain the polynomial coefficients. The method used is similar to that described in Yakimenko (2000) and Ghabcheloo et al. (2009), except that we omit deconfliction of trajectories. In our problem we also assume the searcher travels at constant velocity, so the searcher's acceleration is zero over the entire planning horizon. The description of the method that follows is similar to that found in Yakimenko (2000) and Ghabcheloo et al. (2009), but is included here for

the sake of completeness. We denote by $x(\tau) = (x_1(\tau), x_2(\tau))^T$ the desired path followed by the searcher, parameterized by the virtual arc $\tau \in [0, \tau_f]$, where τ_f is the total virtual arc length between the initial and final positions of the searcher. We represent the state as a function of time using the same notation as the state as a function of the virtual arc length. The meaning should be clear from the context. We represent the coordinates x_1 and x_2 by algebraic polynomials of degree d given by

$$x_j(\tau) = \sum_{i=0}^{d} a_j^i \tau^i, \ j = 1, 2, \tag{V.28}$$

where τ^i indicates that τ is raised to the i^{th} power. The degree d of the polynomials $x_j(\tau)$, $j = 1, 2$ is related to the number of boundary conditions that must be satisfied. In the formulation that follows, we use the prime sign $'$ to indicate $\partial/\partial\tau$, $''$ to indicate the second derivative operator, and $'''$ to indicate the third derivative operator. As before, we use dot symbols above variables to indicate derivatives with respect to time, t. Given τ_f and terminal constraints $x_j(0)$, $x_j'(0)$, $x_j''(0)$, $x_j'''(0)$, $x_j(\tau_f)$, $x_j'(\tau_f)$, $x_j''(\tau_f)$, and $x_j'''(\tau_f)$, for $j = 1, 2$, the coefficients of the seventh-order polynomial can be computed from

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & \tau_f & \tau_f^2 & \tau_f^3 & \tau_f^4 & \tau_f^5 & \tau_f^6 & \tau_f^7 \\ 0 & 1 & 2\tau_f & 3\tau_f^2 & 4\tau_f^3 & 5\tau_f^4 & 6\tau_f^5 & 7\tau_f^6 \\ 0 & 0 & 2 & 6\tau_f & 12\tau_f^2 & 20\tau_f^3 & 30\tau_f^4 & 42\tau_f^5 \\ 0 & 0 & 0 & 6 & 24\tau_f & 60\tau_f^2 & 120\tau_f^3 & 210\tau_f^4 \end{pmatrix} \begin{pmatrix} a_j^0 \\ a_j^1 \\ a_j^2 \\ a_j^3 \\ a_j^4 \\ a_j^5 \\ a_j^6 \\ a_j^7 \end{pmatrix} = \begin{pmatrix} x_j(0) \\ x_j'(0) \\ x_j''(0) \\ x_j'''(0) \\ x_j(\tau_f) \\ x_j'(\tau_f) \\ x_j''(\tau_f) \\ x_j'''(\tau_f) \end{pmatrix}, j = 1, 2 \tag{V.29}$$

In order to express the relationship between τ and t, we define $\lambda(\tau) = \frac{d\tau}{dt}$. Then the temporal and spatial derivatives of x satisfy

$$\dot{x} = \lambda x'. \tag{V.30}$$

As in Yakimenko (2000) and Ghabcheloo et al. (2009) we choose λ to be an affine function of τ, that is, $\lambda(\tau) = \lambda_0 + \frac{\lambda_f - \lambda_0}{\tau_f}\tau$, with $\lambda_0 = \|\dot{x}(0)\|$, and $\lambda_f = \|\dot{x}(1)\|$. By integrating $\dot{\tau} = \lambda(\tau)$, the virtual arc τ and time t are related by the following equations

$$\tau_f = \begin{cases} \lambda_0, & \text{if } \lambda_f = \lambda_0 \\ \frac{\lambda_f - \lambda_0}{\log(\lambda_f/\lambda_0)}, & \text{if } \lambda_f \neq \lambda_0 \end{cases} \tag{V.31}$$

$$\frac{\tau}{\tau_f} = \begin{cases} t, & \text{if } \lambda_f = \lambda_0 \\ \frac{\lambda_0}{\lambda_f - \lambda_0}\left(\left(\frac{\lambda_f}{\lambda_0}\right)^t - 1\right), & \text{if } \lambda_f \neq \lambda_0 \end{cases} \tag{V.32}$$

The second polynomial based method solves the following modified version of (GTP^c), where Table 12 gives a summary of the relationship between the boundary conditions needed to solve (V.29) and the decision vector b.

Table 12. Relationship between decision vector and boundary conditions for indirect polynomial method.

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$x_1(\tau_f)$	$x_2(\tau_f)$	$x_1'(\tau_f)$	$x_2'(\tau_f)$	$x_1'''(0)$	$x_2'''(0)$	$x_1'''(\tau_f)$	$x_2'''(\tau_f)$

$$(GTP^{c,indir}):$$

$$\min_b \left\{ \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} W_{i,j} \exp\left(-\frac{1}{N}\sum_{\gamma=0}^{N-1}\sum_{l=1}^{L} r^l\left(x\left(\frac{\gamma}{N}\right), y^l\left(\frac{\gamma}{N};\alpha_{i,j}\right)\right)\right)\phi(\alpha_{i,j})\right\}$$

$$\begin{aligned} s.t.\ x_j(\tau) &= \sum_{i=0}^{d} a_j^i \tau^i,\ \ j = 1,2, \forall \tau \in [0,\tau_f] \\ v - 0.05v &\leq \lambda(\tau)\,\|x'(\tau)\| \leq v + 0.05v, \forall \tau \in [0,\tau_f] \end{aligned}$$

In problem $(GTP^{c,indir})$, $W_{i,j}$ are the weights for Simpson's rule, $\alpha_{i,j}$ are the discretization points at which the integrand is evaluated, d is the order of the polynomials, and v is the searcher's speed. The total arc length τ_f is computed using (V.31) and the spatial paths $x(\tau)$ and speed profiles $\dot{x}$ are given by (V.29) and (V.30), respectively.

In order to discretize τ, we set $t = \frac{\gamma}{N}$, $\gamma = 0, 1, ..., N-1$ in (V.32). We note that for $j = 1, 2$, $x_j(0)$ are the starting coordinates for the searcher, $x''_j(0) = x''_j(\tau_f) = 0$, and $x'_j(0)$ are determined by solving the initialization problem described in Algorithm V.6.

We then use the following algorithm to solve $(GTP^{c,indir})$.

Algorithm V.6. Approximately solves $(GTP^{c,indir})$.

Data: $N_0 \in \mathcal{N}, M_0 \in \mathbb{N}_3 \times \mathbb{N}_3, \eta_0 \in \mathbf{H}_{c,N_0}$.

Parameter: $d \in \mathbb{N}, d \geq 1$.

Step 0. Set $N = N_0$, $M = M_0$, and $\bar{\eta}_0 = W_N(\eta_0)$.

Step 1. Generate $\{\bar{\eta}_i\}_{i=0}^n$ using $\bar{\eta}_{i+1} \in \mathcal{A}((\overline{GTP}^c_{NM}), \bar{\eta}_i)$, where n is the first iterate satisfying a given optimality tolerance.

Step 2. Compute x^1 from $\bar{\eta}_n$.

Step 3. Set $b_j = x_j(1)$, $j = 1, 2$, $b_3 = \cos(x_3(1))$, $b_4 = \sin(x_3(1))$, and $b_j = 0$, $j = 5, 6, 7, 8$.

Step 4. Generate $\{b_i\}_{i=0}^\infty$ using $b_{i+1} \in \mathcal{A}((GTP^{c,indir}), b_i)$. □

In Step 1 of Algorithm V.6 we are using Algorithm V.2 to solve $(\overline{GTP}^c_{NM})$, and stopping it after satisfying a given optimality tolerance. In Step 2 of Algorithm V.6 we use the solution, $\bar{\eta}_n$, from Step 1 to compute the searcher's trajectory. In Step 3 of Algorithm V.6 we set $b_j = x_j(1)$, $j = 1, 2$, where $(x_1(1), x_2(1))$ is the final position of the searcher we calculated in Step 2. We also use $\bar{\eta}_n$ from Step 2 to determine the searcher's final heading, which we resolve into components to obtain values for b_3 and b_4. We initialize the remaining components of b to zero and move on to Step 4, where we determine an optimized value for b.

b. Problem Instances

All of the problem instances are based on a HVU operating in a 70 nm by 70 nm area of interest using a receding horizon approach, and a one hour planning horizon. For these instances, the HVU is under threat of attack from $L = 10$ targets. The target trajectories are conditioned and generated as described above for ProbA.

The parameter values used to generate the target trajectories are given in Table 6. We again assume that the separation in starting position between members of the swarm is 1.5 nm. We assume that the HVU has a single helicopter escort, whose goal is to minimize the probability that it fails to detect any of the targets during the planning horizon $[0, 1]$. With the exception of $x_1(0)$ and $x_2(0)$, which are given in Table 13, the searcher and detection rate parameters are the same as those given in Tables 7 and 8, respectively, with $K = 1$.

We randomly generate the starting position for the searcher, given in columns two and three of Table 13. The $x_1(0)$ value was selected from the range $[20, 50]$, and the $x_2(0)$ value was selected from the range $[0, 30]$. We let the vertical starting position for the targets on the side (or sides) of the area of interest be a continuum between $[0, 140]$ and again randomly select the lower and upper bounds for the starting position of the targets. The generated values are given in columns four and five of Table 13.

Table 13. Real-time problem instances. For all instances, problem class is (GTP^c), $K = 1$, and $L = 10$.

Instance	$x_1(0)$	$x_2(0)$	Target LB	Target UB	Side Scenario
ProbD	34.56	24.01	19.86	59.05	LHS
ProbE	47.47	11.77	23.97	91.77	Both
ProbF	23.81	27.4	88.53	134.05	RHS

c. ***Numerical Results***

In this section, we provide numerical results obtained using Algorithms V.2, V.5, and V.6 to solve the problem instances given in Table 13. We implement and run all Algorithms in MATLAB 7.11.0 on the same PC described in Section V.B.1, again using SNOPT with default major and minor optimality tolerance as the stopping criteria except as noted below. We separate the results for each problem instance into two tables, one based on the fixed discretization method given in Algorithm V.2, and one based on the polynomial methods given in Algorithms V.5 and V.6. We also provide a figure for each problem instance that shows the resulting trajectories from

the three solution methods. The result from Algorithm V.2 in Figures 12, 13, and 14 uses the first set of parameters that achieve an objective function value smaller than either of the objective function values achieved by Algorithms V.5 and V.6. We plot all results using a value of $N = 320$ and $M = (25, 25)$ to ensure a common standard. This need to "level the playing field" creates two different issues based on the solution methodologies. The results given in Figures 12, 13, and 14 are from solutions generated with N values smaller than 320. As a result, the control inputs for the searcher resulting from Algorithm V.2 are linearly interpolated to $N = 320$ to generate the plots. Algorithms V.5 and V.6 generate polynomial coefficients, so there is no issue with the $N = 320$ used for plotting being larger than the N value used to generate the coefficients. There is an issue, however, when it comes to ensuring that the searcher maintains a constant speed equal to v along its trajectory. We generate the plots for Algorithms V.5 and V.6 using the path for the searcher defined by the polynomial coefficients, but we ensure that the distance traveled between each of the $N = 320$ searcher waypoints is limited to $\frac{v}{320}$, where v is the velocity of the searcher.

Tables 14, 16, and 18 provide results from Algorithm V.2. The initial control input for the searcher is the zero vector of appropriate length, and the initial heading for the searcher is chosen based on the side scenario. For the left hand side and both side scenarios, the initial heading for the searcher is $3\pi/4$. For the right hand side scenario, the initial heading for the searcher is $\pi/4$. Tables 14, 16, and 18 list the values of N and M used to run Algorithm V.2, but all of the objective function values in row three of these tables are evaluated using a common discretization level of $N = 320$ and $M = (25, 25)$. Row four in Tables 14, 16, and 18 gives the time required for the optimization only, and does not include time to build the target trajectories.

To implement Algorithm V.5, we use $d = 7$. The initial input for the polynomial coefficients is chosen based on the side scenario. For the left hand side scenario, all of the coefficients are zero with the exception of a_1^1 which is set to a value of $-x_1(0)$. This results in an initial horizontal line trajectory for the searcher from its

starting location of $(x_1(0), x_2(0))$ and ending at $(0, 24.01)$. For the both side scenario, we set $a_1^1 = -88.98$, $a_1^3 = -0.37$, $a_1^4 = 0.03$, $a_2^1 = 45.64$, $a_2^3 = -0.40$, $a_2^4 = 0.04$, and all the other coefficients equal to zero. This results in an initial straight line trajectory for the searcher with slope -0.507, beginning at $(x_1(0), x_2(0))$ and ending at $(-41.85, 57.05)$. For the right hand side scenario, all of the coefficients are zero with the exception of a_1^1 which is set to a value of $70 - x_1(0) = 46.19$. This results in an initial horizontal line trajectory for the searcher from its starting location of $(x_1(0), x_2(0))$ and ending at $(70, 27.4)$. Tables 15, 17, and 19 list the values of N and M used to run Algorithm V.5, but all of the objective function values in row four of these tables are evaluated using a common discretization level of $N = 320$ and $M = (25, 25)$. Row five in Tables 15, 17, and 19 gives the time required for the optimization only, and does not include time to build the target trajectories. We note that for ProbD the stopping criteria for Algorithm V.5 is not the default major and minor optimality criteria for SNOPT. Because we are interested in real-time methods, we also set a maximum number of major iterations. In the case of ProbD, SNOPT reaches the 700 major iteration limit prior to satisfying its default optimality criteria. This is the reason why the searcher trajectory shown in middle plot of Figure 12 goes so far to the left, beyond the target trajectories. The algorithm was stopped before it could finish optimizing the trajectory to include as much probability "mass" as possible, while still satisfying the velocity constraints.

To implement Algorithm V.6, we use $d = 7$. Because Algorithm V.6 is effectively initialized by Algorithm V.2, we again use the zero vector of appropriate length as the initial control input for the searcher. For the left hand side and both side scenarios, the initial heading for the searcher is $3\pi/4$. For the right hand side scenario, the initial heading for the searcher is $\pi/4$. Tables 15, 17, and 19 list the values of N and M used to run Algorithm V.6, but all of the objective function values in row four of these tables are evaluated using a common discretization level of $N = 320$ and $M = (25, 25)$. Row five in Tables 15, 17, and 19 gives the time required

for the initialization and optimization only, and does not include time to build the target trajectories. We note that the stopping criteria for Algorithm V.6 is not the default major and minor optimality criteria for SNOPT. Again, based on our desire to find real-time methods we set the stopping criteria as 15 major iterations. This is the reason why the searcher trajectories shown in the right plots of Figures 12 and 14 go so far beyond the target trajectories. The algorithm was stopped before it could finish optimizing the trajectories to include as much probability "mass" as possible, while still satisfying the velocity constraints. We note that due to the complicated relationship between the objective function and the decision vector, we use finite differences in SNOPT to estimate the gradients. Finally, we note that we initialize ProbE using $N = 20$ and $M = (9, 9)$ and not $N = 10$ and $M = (5, 5)$. This is because this method is highly dependent on a good initial input for the searcher's final position, and the lower levels of discretization do not provide a solution where the searcher goes to both sides of the AOI to look for the targets.

Table 14. Algorithm V.2 results for ProbD.

N	10	15	20	25	30	35	40	320
M	(5,5)	(7,7)	(9,9)	(11,11)	(13,13)	(15,15)	(15,15)	(25,25)
Obj.	0.0476	0.0294	0.0324	0.0206	0.0097	0.0118	0.0092	0.0088
Time (sec.)	10.67	48.76	141.77	203.48	361.53	612.03	884.68	16223.48

Table 15. Algorithm V.5 and V.6 results for ProbD.

	Direct	Indirect
N	10	10
M	(5,5)	(5,5)
Obj.	0.0246	0.0927
Time (sec.)	317.40	176.50

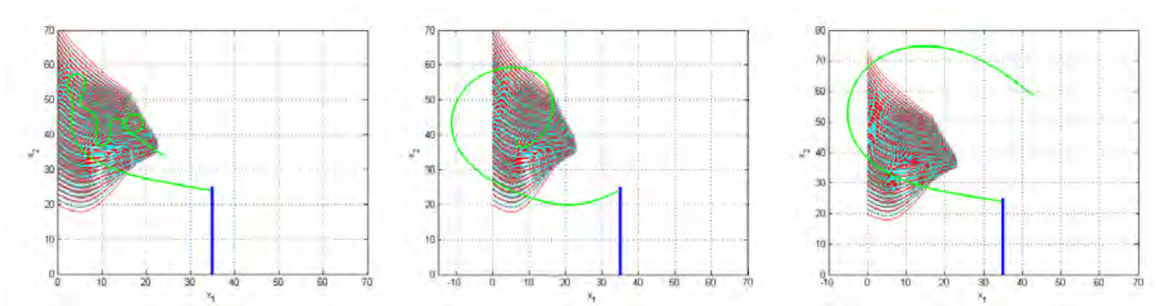

Figure 12. Trajectories for ProbD based on Algorithms V.2, V.5, and V.6. Left: After 203 seconds, Algorithm V.2, $N = 25$, $M = (11, 11)$ for solution, $N = 320$, $M = (25, 25)$ for plot. Middle: After 317 seconds, Algorithm V.5, $N = 320$, $M = (25, 25)$. Right: After 176 seconds, Algorithm V.6, $N = 320$, $M = (25, 25)$.

Table 16. Algorithm V.2 results for ProbE.

N	10	15	20	25	30	35	40	320
M	(5,5)	(7,7)	(9,9)	(11,11)	(13,13)	(15,15)	(15,15)	(25,25)
Obj.	0.5161	0.5022	0.2767	0.2715	0.2715	0.2622	0.2593	0.2555
Time (sec.)	8.29	32.60	52.51	93.55	175.11	284.54	431.95	43000.29

Table 17. Algorithm V.5 and V.6 results for ProbE.

	Direct	Indirect
N	10	20
M	(5,5)	(9,9)
Obj.	0.3526	0.2918
Time (sec.)	66.05	750.00

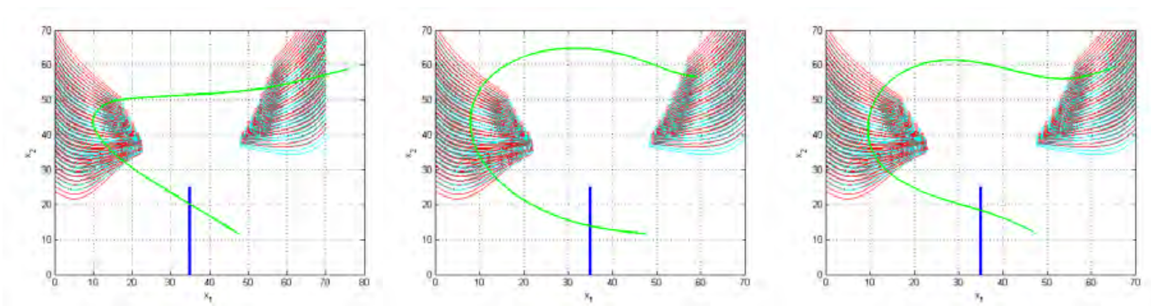

Figure 13. Trajectories for ProbE based on Algorithms V.2, V.5, and V.6. Left: After 53 seconds, Algorithm V.2, $N = 20$, $M = (9, 9)$ for solution, $N = 320$, $M = (25, 25)$ for plot. Middle: After 66 seconds, Algorithm V.5, $N = 320$, $M = (25, 25)$. Right: After 750 seconds, Algorithm V.6, $N = 320$, $M = (25, 25)$.

Table 18. Algorithm V.2 results for ProbF.

N	10	15	20	25	30	35	40	320
M	(5,5)	(7,7)	(9,9)	(11,11)	(13,13)	(15,15)	(15,15)	(25,25)
Obj.	0.0553	0.0330	0.0246	0.0265	0.0249	0.0237	0.0234	0.0216
Time (sec.)	9.29	39.40	74.77	225.61	464.91	818.13	1008.03	35608.20

Table 19. Algorithm V.5 and V.6 results for ProbF.

	Direct	Indirect
N	10	10
M	(5,5)	(5,5)
Obj.	0.0734	0.0421
Time (sec.)	67.85	185.19

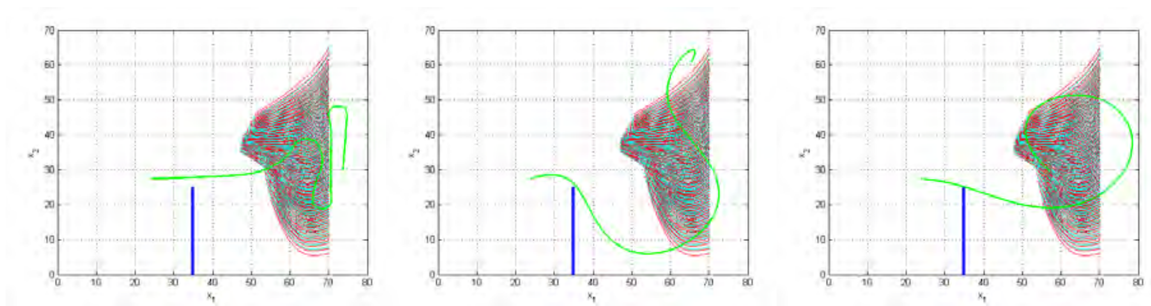

Figure 14. Trajectories for ProbF based on Algorithms V.2, V.5, and V.6. Left: After 39 seconds, Algorithm V.2, $N = 15$, $M = (7, 7)$ for solution, $N = 320$, $M = (25, 25)$ for plot. Middle: After 68 seconds, Algorithm V.5, $N = 320$, $M = (25, 25)$. Right: After 185 seconds, Algorithm V.6, $N = 320$, $M = (25, 25)$.

d. Conclusions

Based on the results presented in this section, it appears that the method which is best suited to provide real-time solutions is Algorithm V.2. The fixed-precision method implemented in Algorithm V.2 consistently provides solutions with the best objective values and the lowest computational times. Table 20 summarizes the comparison between the different methods. To compute the values in Table 20, we assume that the objective value obtained from Algorithm V.2 with $N = 320$ and $M = (25, 25)$ is the "correct" value. For ProbD, we use the objective values from Algorithm V.2 for $N = 25$ and $M = (11, 11)$ and $N = 20$ and $M = (9, 9)$ to compare with Algorithms V.5 and V.6, respectively. For ProbE, we use the objective value from Algorithm V.2 for $N = 20$ and $M = (9, 9)$ to compare with Algorithms V.5 and V.6. For ProbF, we use the objective value from Algorithm V.2 for $N = 15$ and $M = (7, 7)$ to compare with Algorithms V.5 and V.6. The Δt columns in Table 20 give the additional time in seconds necessary for the Algorithms V.5 and V.6 to compute their respective solutions. In fairness, it should again be stressed that the implementation of Algorithm V.6 utilized finite differences to estimate the required gradients. This puts Algorithm V.6 at a disadvantage over the others, although we attempt to mitigate this effect by stopping the algorithm after only 15 major iterations. Future studies should be done with the gradients computed explicitly to ensure a more balanced comparison.

Table 20. Comparison of real-time methods.

Algo.	ProbD % Error	Δt	ProbE % Error	Δt	ProbF % Error	Δt
V.2	134.8%/269.3%	N/A	8.29%	N/A	53.10%	N/A
V.5	180.90%	113.92	37.98%	13.55	239.99%	28.46
V.6	957.20%	34.73	14.19%	697.49	94.92%	145.8

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VI. CONCLUSIONS AND FUTURE WORK

A. CONCLUSIONS

Continuous time-and-space search problems arise in numerous applications such as anti-submarine warfare, search and rescue operations, and protection of HVU's from small boat attack. We consider the problem of detecting targets that seek to harm a HVU, and formulate it as a generalized optimal control problem. While the theory that we develop allows for more general cases, we assume that the targets follow deterministic trajectories, given realizations of random variables that provide information about their initial conditions. We consider two classes of problems, (GTP) and (ITP), based on whether the targets act in a coordinated or independent manner, respectively. For both classes, we consider objective functions that minimize the probability that all of the searchers fail to detect any of the targets during the planning horizon and maximize the expected number of targets detected during the planning horizon.

We develop discretization schemes to solve (GTP) and (ITP) problems, and show that the finite dimensional problems are consistent approximations to their infinite dimensional counterparts. We then use these discretization schemes to develop implementable algorithms that can be used to solve the problems (GTP), (GTP^c), (GTP^e), $(GTP^{c,e})$, (ITP^p), $(ITP^{c,p})$, (ITP^e), and $(ITP^{c,e})$. We provide numerical examples to show that our algorithms are tractable for as many as three searchers and ten targets. For (GTP^c) we compare an algorithm based on fixed discretization schemes to an algorithm based on adaptive discretization schemes. Our results indicate that both methods produce equivalent solutions, but the adaptive discretization schemes require less computational time to acheive them.

We compare three heuristic approaches that have the potential to be used as real-time solution methods onboard unmanned vehicles. We use two different polynomial based approaches, and one fixed-precision method to solve three randomly

generated problem instances. For these problem instances, the fixed-precision method offers the best overall performance. It consistently provides solutions with the best objective values and the lowest computational times.

We also develop rate of convergence results for approximations to infinite dimensional problems that involve a single discretization parameter for the space of decision variables and two discretization parameters for the objective function as a computational budget b tends to infinity. We note that the generalized optimal control problems we define in Chapter III are examples of these infinite dimensional problems. We show that superlinear and linear algorithms can achieve the same theoretical rate of convergence as that achieved by the "ideal" case of a finitely convergent algorithm. We also identify specific optimal discretization policies for both the superlinear and linear cases that achieve this best possible rate. Our analysis indicates that if a linear or superlinear optimization algorithm is used to solve the finite dimensional optimal control problem, with Euler's method used to numerically solve the differential equations and Simpson's rule used to numerically approximate the spatial integration, then the best possible rate of convergence is $b^{-2/3}$. If a second-order Runge-Kutta method is used instead of Euler's method to numerically solve the differential equations, then the best possible rate of convergence is b^{-1}. If a fourth-order Runge-Kutta method is used instead of Euler's method to numerically solve the differential equations, then the best possible rate of convergence is $b^{-4/3}$. Finally, if an ideal method is used instead of Euler's method to numerically solve the differential equations, then the best possible rate of convergence is b^{-2}. If "ideal" methods are used to solve the differential equations as well as evaluate the spatial integration, there is no benefit associated with increasing the discretization parameters N or M, so $b = n$, where n is the number of iterations of the optimization algorithm. The resulting asymptotic rates for a superlinear optimization algorithm with order $\gamma \in (1, \infty)$ and $c \in (0, 1)$, and a linear optimization algorithm with rate constant $\bar{c} \in (0, 1)$ are c^{γ^b} and $\bar{c}^b$, respectively. Based on our analysis, it appears that it is possible to improve the run-

time performance of our algorithms if a higher-order method such as Runge-Kutta or pseudo-spectral is used in place of Euler's method to numerically solve the differential equations.

B. FUTURE WORK

This dissertation suggests a number of possibilities for extension and future work. The first area we consider for additional work is the development of a real-time solution method. As discussed in Section V.B.3.d, the implementation of Algorithm V.6 used finite differences to estimate the required gradients. Future work should include explicit computation of all required derivatives, so that the implementation of Algorithm V.6 is on a more equal footing with those of the other two potential real-time methods. An opportunity for extension would be to use something other than algebraic polynomials as the basis for the searcher trajectories in Algorithms V.5 and V.6. One suggestion would be to use Chebyshev polynomials. In Caporale & Cerrato (2008) Chebyshev polynomials in conjunction with Chebyshev nodes were shown to provide excellent approximations to the solutions of linear partial differential equations, indicating that there might be potential for their use in Algorithms V.5 or V.6. Another suggestion would be to use Bézier curves in Algorithm V.6. Because Bézier curves are completely contained in the convex hull of their control points, they would be particularly well suited to an indirect curve fitting method where the control points could be determined based on the low level initialization as in Algorithm V.6.

As discussed in Section VI.A above, there is potential for run-time improvement if higher order methods are used in place of Euler's method to numerically solve the differential equations. The consistent approximation theory and implementable algorithms developed in Chapters III and V, respectively, could be extended based on using Runge-Kutta or pseudo-spectral methods instead of Euler's method to numerically solve the differential equations, however, we foresee numerous technical challenges.

In this dissertation, we consider two different types of objective functions. Another natural extension would be to consider additional types of objective functions. In the sections that follow, we present two alternative objective functions that, while potentially more difficult to solve, may be of operational interest in many situations.

1. Minimize Expected Time Until First Detection

We now consider a new performance metric for our optimization problem that is based on minimizing the expected time until the first detection of all the targets. In Chapter II, we derived (II.3), which is an expression for the probability that the k^{th} searcher does not detect the l^{th} target during $[0,t]$, $t \in [0,1]$ We define the random variable T as the time of first detection of the l^{th} target, and note that

$$P\left(\{T > t\}\right) = q^{k,l}(t;\alpha) = \exp\left(-\int_0^t r^{k,l}\left(x^k(s), y^l(s;\alpha)\right) ds\right) = 1 - F(t), \quad \text{(VI.1)}$$

where $F(t)$ is the cumulative distribution function for the random variable T. Then for the case of one searcher and one target the conditional expected time of first detection given a particular target trajectory is given by

$$E[T|\alpha] = \int_0^1 t f_T(t) dt, \quad \text{(VI.2)}$$

where $f_T(t)$ is the probability density function of the random variable T. We note that the support of the random variable T is $[0,1]$, so $E[T|\alpha]$ is finite. We also note that we can use (VI.1) and integration by parts to show that (VI.2) can be re-written in terms of $q^{k,l}(t;\alpha)$ as follows

$$\begin{aligned} E[T|\alpha] = \int_0^1 q^{k,l}(t;\alpha)dt &= \int_0^1 \exp\left(-\int_0^t r^{k,l}\left(x^k(s), y^l(s;\alpha)\right) ds\right) dt \\ &= \int_0^1 P\left(\{T > t\}\right) dt = P\left(\{T > t\}\right) t\Big|_0^1 - \int_0^1 t\left(-f_T(t)\right) dt \\ &= \int_0^1 t f_T(t) dt. \end{aligned} \quad \text{(VI.3)}$$

If we assume that the searchers make independent detection attempts, then for the

case of multiple searchers the conditional expected time of first detection given a particular target trajectory is given by

$$E[T|\alpha] = \int_0^1 \prod_{k=1}^{K} \exp\left(-\int_0^t r^{k,l}\left(x^k(s), y^l(s;\alpha)\right) ds\right) dt \tag{VI.4}$$

$$= \int_0^1 \exp\left(-\sum_{k=1}^{K}\int_0^t r^{k,l}\left(x^k(s), y^l(s;\alpha)\right) ds\right) dt \tag{VI.5}$$

$$= \int_0^1 \exp\left(-\int_0^t \sum_{k=1}^{K} r^{k,l}\left(x^k(s), y^l(s;\alpha)\right) ds\right) dt. \tag{VI.6}$$

If we assume that the random variables that the target motion is conditioned upon are independent across targets, then the expected time of first detection of the l^{th} target is given by

$$E[T] = \int_{\alpha^l \in A} \left[\int_0^1 \exp\left(-\int_0^t \sum_{k=1}^{K} r^{k,l}\left(x^k(s), y^l(s;\alpha^l)\right) ds\right) dt\right] \phi^l\left(\alpha^l\right) d\alpha^l. \tag{VI.7}$$

2. Herding Formulation

Throughout this dissertation, we have focused on detecting the potential threats to the HVU. Once the potential threats are located, the searchers could become defenders who now select controls such that the potential attackers are herded away from the HVU. In a herding model, we assume a different target motion model where the targets' motion is defined by coupled dynamics, which we describe in detail below. Then, a natural metric for the degree of success in herding would be to maximize the minimum distance between the HVU and any attacker at any instance of time in the planning horizon. This herding success metric is given by

$$\max_{x^k(\cdot)}\left\{\min_{l,t}\int_{\alpha^l \in A}\left[\left(x_1^0(t) - y_1^l\left(t;\alpha^l\right)\right)^2 + \left(x_2^0(t) - y_2^l\left(t;\alpha^l\right)\right)^2\right]\phi^l\left(\alpha^l\right) d\alpha^l\right\}. \tag{VI.8}$$

In order to formulate the dynamics between the defenders and the attackers, we first define some necessary quantities. First, we define

$$range_{ad}^{l,k}(t) = \sqrt{\left(x_1^k(t) - y_1^l\left(t;\alpha^l\right)\right)^2 + \left(x_2^k(t) - y_2^l\left(t;\alpha^l\right)\right)^2}, \tag{VI.9}$$

which gives the range between the l^{th} attacker and the k^{th} defender at time t, given a particular attacker trajectory. We also define

$$range^l_{HVU}(t) = \sqrt{\left(x^0_1(t) - y^l_1\left(t;\alpha^l\right)\right)^2 + \left(x^0_2(t) - y^l_2\left(t;\alpha^l\right)\right)^2}, \tag{VI.10}$$

which gives the range between the l^{th} attacker and the HVU at time t, given a particular attacker trajectory. Next, we define

$$\theta^{l,k}_{ad}(t) = \tan^{-1}\left(\frac{x^k_2(t) - y^l_2\left(t;\alpha^l\right)}{x^k_1(t) - y^l_1\left(t;\alpha^l\right)}\right), \tag{VI.11}$$

which gives the angle between the l^{th} attacker and the k^{th} defender at time t, given a particular attacker trajectory. Finally, we define

$$\theta^l_{HVU}(t) = \tan^{-1}\left(\frac{x^0_2(t) - y^l_2\left(t;\alpha^l\right)}{x^0_1(t) - y^l_1\left(t;\alpha^l\right)}\right), \tag{VI.12}$$

which gives the angle between the l^{th} attacker and the HVU at time t, given a particular attacker trajectory. Then, the dynamics of the l^{th} attacker given a particular realization of the random variable α are given by

$$\dot{y}^l_1(t) = -w^l_1 \sum_{k=1}^{K} \frac{\cos\theta^{l,k}_{ad}(t)}{\left[range^{l,k}_{ad}(t)\right]^2} + w^l_2 \frac{\cos\theta^l_{HVU}(t)}{\left[range^l_{HVU}(t)\right]^2} \tag{VI.13}$$

$$\dot{y}^l_2(t) = -w^l_1 \sum_{k=1}^{K} \frac{\sin\theta^{l,k}_{ad}(t)}{\left[range^{l,k}_{ad}(t)\right]^2} + w^l_2 \frac{\sin\theta^l_{HVU}(t)}{\left[range^l_{HVU}(t)\right]^2}, \tag{VI.14}$$

where w^l_1 and w^l_2 are user specified weights that define the relative importance of the two components of the attacker dynamics. The component associated with w^l_1 is due to the herding effects of the defenders. In a manner similar to the approach used in Lu (2006), we assume that the l^{th} attacker moves in a straight line away (hence the minus sign in front of w^l_1) from the weighted sum of "influences" of the k defenders, and that the l^{th} attacker's velocity is bounded by the inverse of the distance between it and the k^{th} defender squared. The component associated with w^l_2 is due to the attacker's desire to get close to the HVU in order to facilitate his attack. Again, we

assume that the l^{th} attacker moves in a straight line towards (hence the plus sign in front of w_2^l) the HVU, and that the l^{th} attacker's velocity is bounded by the inverse of the distance between it and the HVU squared.

One way that w_1^l and w_2^l could be specified would be to define a distance called $panic^l$ that gives the range from the HVU that the l^{th} attacker begins to worry about being detected by the defenders. In this case w_1^l and w_2^l would be given by

$$w_1^l = \frac{panic^l}{range_{HVU}^l} \tag{VI.15}$$

$$w_2^l = \frac{range_{HVU}^l}{panic^l}. \tag{VI.16}$$

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VII. APPENDIX: MATHEMATICAL BACKGROUND

Throughout this dissertation, we consider the Lipschitz continuity of functions *relative* to the set $\mathbf{H}$, and the differentiability of functions on the set $\mathbf{H}^0$, *relative* to the set $\mathbf{H}$ as defined in Polak (1997). For the sake of completeness, we include the pertinent definitions from pp. 652–656 of Polak (1997). We begin with the definitions related to continuity.

Definition VII.1. Let $\mathcal{V}$ be a real normed space and let S be a convex subset of $\mathcal{V}$.

(i) A function $f : \mathcal{V} \to \mathbb{R}^m$ is said to be continuous at a point $x \in \mathcal{V}$, if, for every $\delta > 0$, there exists a $\rho > 0$ such that

$$\|f(x') - f(x)\| < \delta, \forall x' \in \overset{\circ}{B}(x, \rho), \tag{VII.1}$$

where $\overset{\circ}{B}(x, \rho) \triangleq \{x' \in \mathbb{R}^n \mid \|x' - x\| < \rho\}$. A function $f : \mathcal{V} \to \mathbb{R}^m$ is said to be continuous (continuous on S) if it is continuous at all $x \in \mathcal{V}$ ($x \in S$).

(ii) A function $f : \mathcal{V} \to \mathbb{R}^m$ is said to be continuous, relative to S (S-continuous), if, for every $x \in S$ and for every $\delta > 0$, there exists a $\rho > 0$ such that

$$\|f(x') - f(x)\| < \delta, \forall x' \in \overset{\circ}{B}(x, \rho) \cap S. \tag{VII.2}$$

□

Next we state definitions related to differentiability.

Definition VII.2. Let $\mathcal{V}$ be a real normed space.

(i) We will say that a continuous function $f : \mathcal{V} \to \mathbb{R}^m$ is Gateaux differentiable at a point $x^* \in \mathcal{V}$, if there exists a bounded linear operator $f_x(x^*) : \mathcal{V} \to \mathbb{R}^m$ such that, for every $\delta x \in \mathcal{V}$,

$$\lim_{\lambda \downarrow 0} \frac{\|f(x^* + \lambda \delta x) - f(x^*) - \lambda f_x(x^*) \delta x\|}{\lambda} = 0. \tag{VII.3}$$

We will call $f_x(x^*)$ the Gateaux derivative of $f(\cdot)$ at x^*.

We will say that a continuous function $f : \mathcal{V} \to \mathbb{R}^m$ is Gateaux differentiable on a subset S of $\mathcal{V}$, if it is Gateaux differentiable at all $x \in S$.

(ii) We will say that a continuous function $f : \mathcal{V} \to \mathbb{R}^m$ is Frechet differentiable at a point $x^* \in \mathcal{V}$, if it is Gateaux differentiable at x^*, with Gateaux derivative $f_x(x^*)$, and, in addition, the Gateaux derivative has the property that

$$\lim_{\delta x \to 0} \frac{\|f(x^* + \delta x) - f(x^*) - f_x(x^*)(\delta x)\|}{\|\delta x\|} = 0. \tag{VII.4}$$

In this case, we will also call $f_x(x^*)$ the Frechet derivative of $f(\cdot)$ at x^*.

We will say that a continuous function $f : \mathcal{V} \to \mathbb{R}^m$ is Frechet differentiable on a subset S of $\mathcal{V}$, if it is Frechet differentiable at all $x \in S$.

(iii) Let $S' \subset S$ be two convex subsets of $\mathcal{V}$. We will say that a continuous function $f : S \to \mathbb{R}^m$ is Gateaux differentiable, relative to S, (Gateaux S-differentiable) at a point $x^* \in S'$, if there exists a bounded linear operator $f_x(x^*) : \mathcal{V} \to \mathbb{R}^m$ such that, for every $\delta x \in \mathcal{V}$ such that $x^* + \lambda^* \delta x \in S$, for some $\lambda^* > 0$,

$$\lim_{\lambda \downarrow 0} \frac{\|f(x^* + \lambda \delta x) - f(x^*) - \lambda f_x(x^*) \delta x\|}{\lambda} = 0. \tag{VII.5}$$

We will call $f_x(x^*)$ the Gateaux S-derivative of $f(\cdot)$ at x^*.

We will say that $f(\cdot)$ is Gateaux S-differentiable on S', if it is Gateaux S-differentiable at all $x \in S'$.

(iv) Let $S' \subset S$ be two convex subsets of $\mathcal{V}$. We will say that a continuous function $f : S \to \mathbb{R}^m$ is Frechet differentiable, relative to S, (S-differentiable) at a point $x^* \in S'$, if it is Gateaux S-differentiable at x^*, with Gateaux S-derivative $f_x(x^*)$, and, in addition, the Gateaux S-derivative has the property that

$$\lim_{\substack{\delta x \to 0 \\ x^* + \delta x \in S}} \frac{\|f(x^* + \delta x) - f(x^*) - f_x(x^*)(\delta x)\|}{\|\delta x\|} = 0. \tag{VII.6}$$

In this case, we will also call $f_x(x^*)$ the Frechet S-derivative of $f(\cdot)$ at x^*.

We will say that $f(\cdot)$ is S-differentiable on S', if it is Frechet S-differentiable at all $x \in S'$.

$\square$

The next two paragraphs are a restatement of the beginning of Section 5.6.2 from Polak (1997), but are included here for the sake of completeness to provide a detailed explanation of what it means for a function to be continuous *relative* to the set $\mathbf{H}$ and Gateaux differentiable at $\eta \in \mathbf{H}^0$, *relative* to the set $\mathbf{H}$.

Based on Definition VII.1, we can now give a detailed explanation of what it means for a function to be continuous *relative* to the set $\mathbf{H}$. Let $\mathbf{S}$ be a convex subset of $\mathbf{H}$. Then, according to Definition VII.1, a function $f : \mathbf{S} \to \mathbb{R}^n$ is continuous *relative* to $\mathbf{H}$, at a point $\eta \in \mathbf{S}$, if, for any sequence $\{\eta_i\}_{i=0}^{\infty}$, with $\eta_i \in \mathbf{H}$ for all $i \in \mathbb{N}$, such that $\eta_i \to \eta$, as $i \to \infty$, $f(\eta_i) \to f(\eta)$, as $i \to \infty$. It should be noted, however, that if $\{\eta_i\}_{i=0}^{\infty}$ is any arbitrary sequence converging to η, then it is possible that $\eta_i \notin \mathbf{H}$ for all $i \in \mathbb{N}$. Then, we cannot claim that $f(\eta_i) \to f(\eta)$, as $i \to \infty$, and hence we cannot necessarily conclude that $f(\cdot)$ is continuous at η. To keep our terminology as concise as possible, as in Definition VII.1, given $\mathbf{S}$, a convex subset of $\mathbf{H}$, we will say that a function $f : \mathbf{S} \to \mathbb{R}^n$ is $\mathbf{H}$-continuous when we mean that it is continuous *relative* to $\mathbf{H}$.

Similarly, based on Definition VII.2 we can give a detailed explanation of what it means for a function to be Gateaux differentiable at $\eta \in \mathbf{H}^0$, *relative* to the set $\mathbf{H}$. We begin by noting that given any $\eta \in \mathbf{H}^0$ and any $\delta\eta \in H_{\infty,2}$, there always exists a $\lambda > 0$ such that $\eta + \lambda\delta\eta \in \mathbf{H}$. Then it follows from Definition VII.2(iii) that a function $f : \mathbf{H} \to \mathbb{R}^n$ is Gateaux differentiable at $\eta \in \mathbf{H}^0$, relative to $\mathbf{H}$, if there exists a continuous linear map $Df(\eta; \cdot)$ from $H_{\infty,2}$ into $\mathbb{R}^n$, such that for all $\delta\eta \in H_{\infty,2}$,

$$\lim_{\lambda \downarrow 0} \frac{f(\eta + \lambda\delta\eta) - f(\eta) - \lambda Df(\eta; \delta\eta)}{\lambda} = 0. \tag{VII.7}$$

As a result, if $f : \mathbf{H} \to \mathbb{R}^n$ is Gateaux differentiable at $\eta \in \mathbf{H}^0$, *relative* to $\mathbf{H}$, it is also Gateaux differentiable at η. It also follows from Definition VII.2(iv) that $f : \mathbf{H} \to \mathbb{R}^n$ is Frechet differentiable relative to $\mathbf{H}$ (Frechet $\mathbf{H}$-differentiable) at $\eta \in \mathbf{H}^0$, if it is Gateaux differentiable at η and the Gateaux differential $Df(\eta; \cdot)$ has the property that

$$\lim_{\substack{\eta' \in \mathbf{H} \\ \|\eta' - \eta\|_{H_2} \to 0}} \frac{\|f(\eta') - f(\eta) - Df(\eta; \eta' - \eta)\|}{\|\eta' - \eta\|_{H_2}} = 0. \tag{VII.8}$$

Then by definition, a function $f : \mathbf{H} \to \mathbb{R}^n$ is Gateaux/Frechet $\mathbf{H}$-differentiable on $\mathbf{H}^0$, if it is Gateaux/Frechet $\mathbf{H}$-differentiable at every $\eta \in \mathbf{H}^0$.

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